Lesson 6: Modeling a Context from Data

Classwork

Opening Exercise

1. Identify the type of function that each table represents (e.g., quadratic, linear, exponential, square root, etc.).

A

x	y
1	5
2	7
3	9
4	11
5	13

B

x	y
1	6
2	9
3	13.5
4	20.25
5	30.375

C

x	y
1	3
2	12
3	27
4	48
5	75

2. Explain how you were able to identify the function.

3. Find the symbolic representation of the function.

4. Plot the graphs of your data.

Mathematics: Standards for Mathematical Practice

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important “processes and proficiencies” with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council’s report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy).

1. Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, “Does this make sense?” They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2. Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

3. Construct viable arguments and critique the reasoning of others.

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4. Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5. Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6. Attend to precision.

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7. Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

8. Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

PARCC Model Content Framework for Mathematics for Algebra I

Algebra I Overview

Numerals in parentheses designate individual content standards that are eligible for assessment in whole or in part. Underlined numerals (e.g., 1) indicate standards eligible for assessment on two or more end-of-course assessments. For more information, see Tables 1 and 2. Course emphases are indicated by: ■ Major Content; □ Supporting Content; ○ Additional Content. Not all CCSSM content standards in a listed domain or cluster are assessed.

The Real Number System (N-RN)

- ○ Use properties of rational and irrational numbers (3)

Quantities★(N-Q)

- □ Reason quantitatively and use units to solve problems (1, 2, 3)

Seeing Structure in Expressions (A-SSE)

- ■ Interpret the structure of expressions (1, 2)
- □ Write expressions in equivalent forms to solve problems (3)

Arithmetic with Polynomials and Rational Expressions (A-APR)

- ■ Perform arithmetic operations on polynomials (1)
- □ Understand the relationship between zeros and factors of polynomials (3)

Creating Equations★ (A-CED)

- ■ Create equations that describe numbers or relationships (1, 2, 3, 4)

Reasoning with Equations and Inequalities (A-REI)

- ■ Understand solving equations as a process of reasoning and explain the reasoning (1)
- ■ Solve equations and inequalities in one variable (3, 4)
- ○ Solve systems of equations (5, 6)
- ■ Represent and solve equations and inequalities graphically (10, 11, 12)

Interpreting Functions (F-IF)

- ■ Understand the concept of a function and use function notation (1, 2, 3)
- ■ Interpret functions that arise in applications in terms of the context (4, 5, 6)
- □ Analyze functions using different representations (7, 8, 9)

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

PARCC

Building Functions (F-BF)

- Build a function that models a relationship between two quantities (1)
- Build new functions from existing functions (3)

Linear, Quadratic, and Exponential Models★ (F-LE)

- Construct and compare linear, quadratic, and exponential models and solve problems (1, 2, 3)
- Interpret expressions for functions in terms of the situation they model (5)

Interpreting categorical and quantitative data (S-ID)

- Summarize, represent, and interpret data on a single count or measurement variable (1, 2, 3)
- Summarize, represent, and interpret data on two categorical and quantitative variables (5, 6)
- Interpret linear models (7, 8, 9)

Examples of Key Advances from Grades K–8

- Having already extended arithmetic from whole numbers to fractions (grades 4-6) and from fractions to rational numbers (grade 7), students in grade 8 encountered particular irrational numbers such as $\sqrt{5}$ or π. In Algebra I, students will begin to understand the real number *system*. For more on the extension of number systems, see page 58 of the standards.

- Students in middle grades worked with measurement units, including units obtained by multiplying and dividing quantities. In Algebra I, students apply these skills in a more sophisticated fashion to solve problems in which reasoning about units adds insight (N-Q).

- Themes beginning in middle school algebra continue and deepen during high school. As early as grades 6 and 7, students began to use the properties of operations to generate equivalent expressions (6.EE.3, 7.EE.1). By grade 7, they began to recognize that rewriting expressions in different forms could be useful in problem solving (7.EE.2). In Algebra I, these aspects of algebra carry forward as students continue to use properties of operations to rewrite expressions, gaining fluency and engaging in what has been called "mindful manipulation."[26]

- Students in grade 8 extended their prior understanding of proportional relationships to begin working with functions, with an emphasis on linear functions. In Algebra I, students will master linear and quadratic functions. Students encounter other kinds of functions to ensure that general principles are perceived in generality, as well as to enrich the range of quantitative relationships considered in problems.

- Students in grade 8 connected their knowledge about proportional relationships, lines, and linear equations (8.EE.5, 6). In Algebra I, students solidify their understanding of the analytic geometry of lines. They understand that in the Cartesian coordinate plane:

  - The graph of any linear equation in two variables is a line.

[26] See, for example, "Mindful Manipulation," in *Focus in High School Mathematics: Reasoning and Sense Making* (National Council of Teachers of Mathematics, 2009).

- Any line is the graph of a linear equation in two variables.

- As students acquire mathematical tools from their study of algebra and functions, they apply these tools in statistical contexts (e.g., S-ID.6). In a modeling context, they might informally fit a quadratic function to a set of data, graphing the data and the model function on the same coordinate axes. They also draw on skills they first learned in middle school to apply basic statistics and simple probability in a modeling context. For example, they might estimate a measure of center or variation and use it as an input for a rough calculation.

- Algebra I techniques open a huge variety of word problems that can be solved that were previously inaccessible or very complex in grades K-8. This expands problem solving from grades K-8 dramatically.

Discussion of Mathematical Practices in Relation to Course Content

Two overarching practices relevant to Algebra I are:

- **Make sense of problems and persevere in solving them** (MP.1).

- **Model with mathematics** (MP.4).

Indeed, other mathematical practices in Algebra I might be seen as contributing specific elements of these two. The intent of the following set is not to decompose the above mathematical practices into component parts but rather to show how the mathematical practices work together.

- **Reason abstractly and quantitatively** (MP.2). This practice standard refers to one of the hallmarks of algebraic reasoning, the process of decontextualization and contextualization. Much of elementary algebra involves creating abstract algebraic models of problems (A-CED, F-BF) and then transforming the models via algebraic calculations (A-SSE, A-APR, F-IF) to reveal properties of the problems.

- **Use appropriate tools strategically** (MP.5). Spreadsheets, a function modeling language, graphing tools, and many other technologies can be used strategically to gain understanding of the ideas expressed by individual content standards and to model with mathematics.

- **Attend to precision** (MP.6). In algebra, the habit of using precise language is not only a mechanism for effective communication but also a tool for understanding and solving problems. Describing an idea precisely (A-CED, A-REI) helps students understand the idea in new ways.

- **Look for and make use of structure** (MP.7). For example, writing $49x^2 + 35x + 6 \text{ as } (7x)^2 + 5(7x) + 6$, a practice many teachers refer to as "chunking,'' highlights the structural similarity between this expression and $z^2 + 5z + 6$, leading to a factorization of the original: $((7x) + 3)((7x) + 2)$ (A-SSE, A-APR).

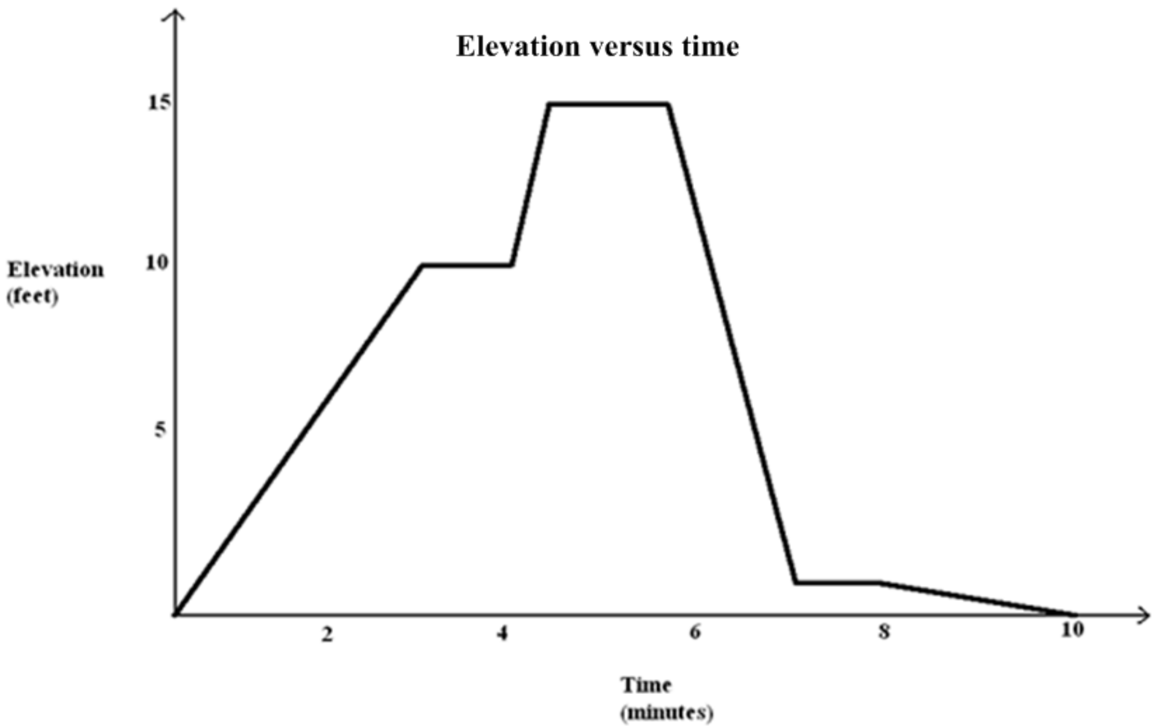
- **Look for and express regularity in repeated reasoning** (MP.8). Creating equations or functions to model situations is harder for many students than working with the resulting expressions. An effective way to help students develop the skill of describing general relationships is to work through several specific examples and then express what they are doing with algebraic symbolism (A-CED). For example, when comparing two different text messaging plans, many students who can compute the cost for a given number of minutes have a hard time writing general formulas that express the cost of each plan for *any* number of minutes. Constructing these formulas can be facilitated by methodically calculating the cost for several different input values and then expressing the steps in the calculation, first in words and then in algebraic symbols. Once such expressions are obtained, students can find the break-even point for the two plans, graph the total cost against the number of messages sent, and make a complete analysis of the two plans.

Fluency Recommendations

A/G Algebra I students become fluent in solving characteristic problems involving the analytic geometry of lines, such as writing down the equation of a line given a point and a slope. Such fluency can support them in solving less routine mathematical problems involving linearity, as well as in modeling linear phenomena (including modeling using systems of linear inequalities in two variables).

A-APR.1 Fluency in adding, subtracting, and multiplying polynomials supports students throughout their work in algebra, as well as in their symbolic work with functions. Manipulation can be more mindful when it is fluent.

A-SSE.1b Fluency in transforming expressions and chunking (seeing parts of an expression as a single object) is essential in factoring, completing the square, and other mindful algebraic calculations.


Elevation versus time
15
10
5
Elevation
(feet)
2
4
6
8
10
Time
(minutes)

Name ______________________________ Date ________________

1. Jacob lives on a street that runs east and west. The grocery store is to the east and the post office is to the west of his house. Both are on the same street as his house. Answer the questions below about the following story:

   At $1{:}00$ p.m., Jacob hops in his car and drives at a constant speed of 25 mph for 6 minutes to the post office. After 10 minutes at the post office, he realizes he is late and drives at a constant speed of 30 mph to the grocery store, arriving at $1{:}28$ p.m. He then spends 20 minutes buying groceries.

   a. Draw a graph that shows the distance Jacob's car is from his house with respect to time. Remember to label your axes with the units you chose and any important points (home, post office, grocery store).

b. On the way to the grocery store, Jacob looks down at his watch and notes the time as he passes his house. What time is it when he passes his house? Explain how you found your answer.

c. If he drives directly back to his house after the grocery story, what was the total distance he traveled to complete his errands? Show how you found your answer.

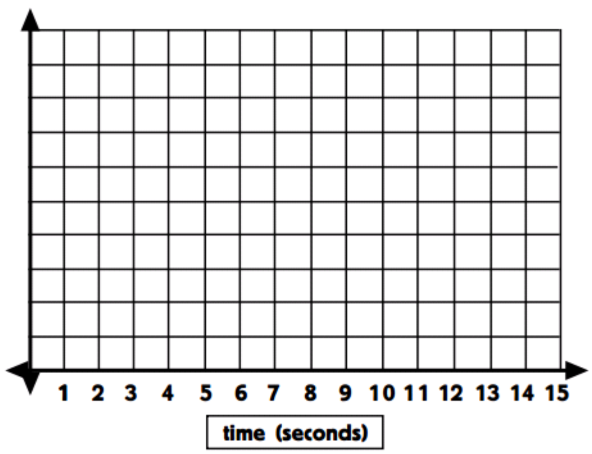
Exploratory Challenge

Watch the following graphing story:

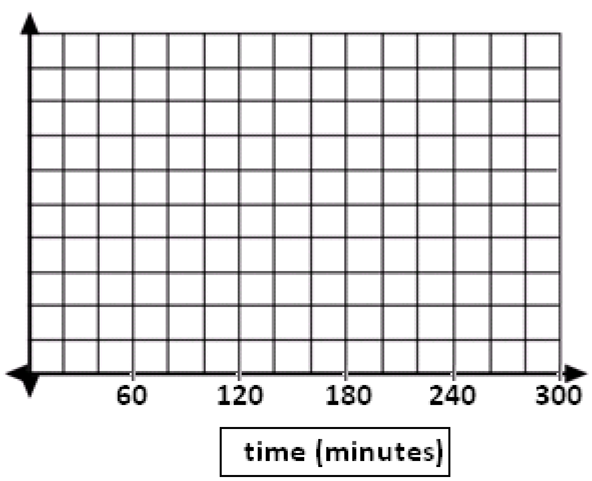
https://www.youtube.com/watch?v=gEwzDydciWc

The video shows bacteria doubling every second.

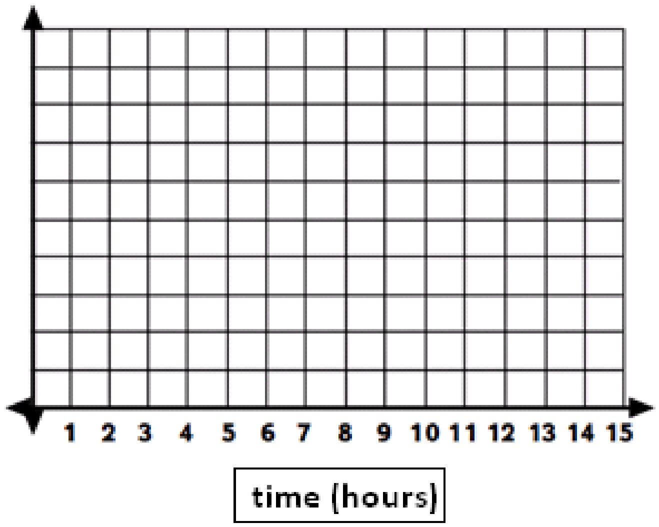
1. Graph the number of bacteria versus time in seconds. Begin by counting the number of bacteria present at each second and plotting the appropriate points on the set of axes below. Consider how you might handle estimating these counts as the population of the bacteria grows.



2. Graph the number of bacteria versus time in minutes.



3. Graph the number of bacteria versus time in hours (for the first five hours).



Lesson 14: Modeling Relationships with a Line

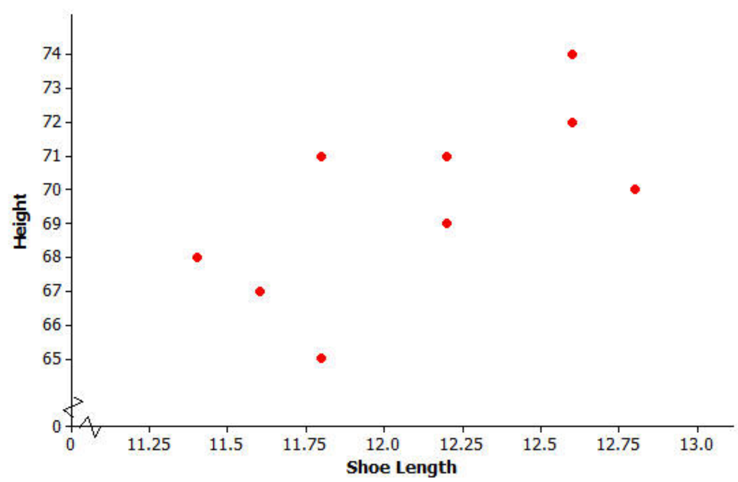
Classwork

Example 1: Using a Line to Describe a Relationship

Kendra likes to watch crime scene investigation shows on television. She watched a show where investigators used a shoe print to help identify a suspect in a case. She questioned how possible it is to predict someone's height is from his shoe print.

To investigate, she collected data on shoe length (in inches) and height (in inches) from 10 adult men. Her data appear in the table and scatter plot below.

x = Shoe Length	*y* = Height
12.6	74
11.8	65
12.2	71
11.6	67
12.2	69
11.4	68
12.8	70
12.2	69
12.6	72
11.8	71



Exercises 1–2

1. Is there a relationship between shoe length and height?

2. How would you describe the relationship? Do the men with longer shoe lengths tend be taller?

Example 2: Using Models to Make Predictions

When two variables x and y are linearly related, you can use a line to describe their relationship. You can also use the equation of the line to predict the value of the y-variable based on the value of the x-variable.

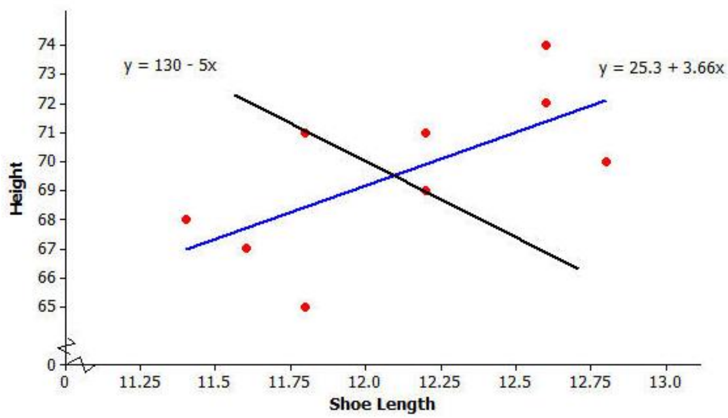
For example, the line $y = 25.3 + 3.66x$ might be used to describe the relationship between shoe length and height, where x represents shoe length and y represents height. To predict the height of a man with a shoe length of 12, you would substitute 12 in for x in the equation of the line and then calculate the value of y:

$$y = 25.3 + 3.66x = 25.3 + 3.66(12) = 69.22$$

You would predict a height of 69.22 inches for a man with a shoe length of 12 inches.

Exercises 3–7

3. Below is a scatter plot of the data with two linear models; $y = 130 - 5x$ and $y = 25.3 + 3.66x$. Which of these two models does a better job of describing how shoe length (x) and height (y) are related? Explain your choice.



4. One of the men in the sample has a shoe length of 11.8 inches and a height of 71 inches. Circle the point in the scatter plot in Question 3 that represents this man.

5. Suppose that you do not know this man's height, but do know that his shoe length is 11.8 inches. If you use the model $y = 25.3 + 3.66x$, what would you predict his height to be? If you use the model $y = 130 - 5x$, what would you predict his height to be?

6. Which model was closer to the actual height of 71 inches? Is that model a better fit to the data? Explain your answer.

7. Is there a better way to decide which of two lines provides a better description of a relationship (rather than just comparing the predicted value to the actual value for one data point in the sample)?

Example 3: Residuals

One way to think about how useful a line is for describing a relationship between two variables is to use the line to predict the y values for the points in the scatter plot. These predicted values could then be compared to the actual y values.

For example, the first data point in the table represents a man with a shoe length of 12.6 inches and height of 74 inches. If you use the line $y = 25.3 + 3.66x$ to predict this man's height, you would get:

$$\begin{aligned} y &= 25.3 + 3.66x \\ &= 25.3 + 3.66(12.6) \\ &= 71.42 \textit{ inches} \end{aligned}$$

Because his actual height was 74 inches, you can calculate the prediction error by subtracting the predicted value from the actual value. This prediction error is called a *residual*. For the first data point, the residual is calculated as follows:

$$\begin{aligned} Residual &= actual\ y\ value - predicted\ y\ value \\ &= 74 - 71.42 \\ &= 2.58 \textit{ inches} \end{aligned}$$

Lesson 6: Algebraic Expressions—The Distributive Property

Problem Set

1. Insert parentheses to make each statement true.
   a. $2 + 3 \times 4^2 + 1 = 81$
   b. $2 + 3 \times 4^2 + 1 = 85$
   c. $2 + 3 \times 4^2 + 1 = 51$
   d. $2 + 3 \times 4^2 + 1 = 53$

2. Using starting symbols of w, q, 2, and -2, which of the following expressions will NOT appear when following the rules of the game played in Exercise 3?
   a. $7w + 3q + (-2)$
   b. $q - 2$
   c. $w - q$
   d. $2w + 6$
   e. $-2w + 2$

3. Luke wants to play the 4-number game with the numbers 1, 2, 3, and 4 and the operations of addition, multiplication, AND subtraction.

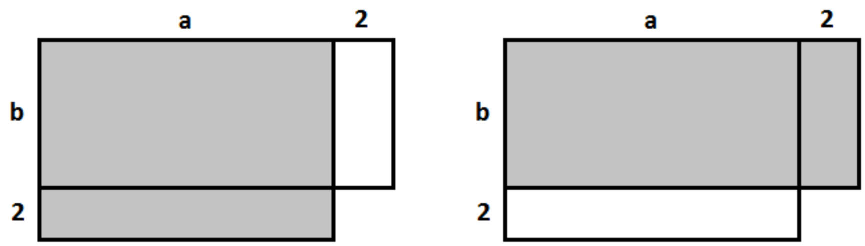
   Leoni responds, "Or we just could play the 4-number game with just the operations of addition and multiplication, but now with the numbers -1, -2, -3, -4, 1, 2, 3, and 4 instead."

   What observation is Leoni trying to point out to Luke?

4. Consider the expression: $(x + 3) \cdot (y + 1) \cdot (x + 2)$.
   a. Draw a picture to represent the expression.
   b. Write an equivalent expression by applying the distributive property.

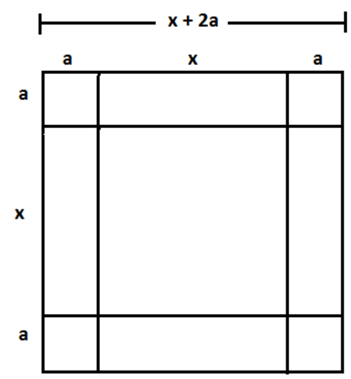
5.

a. Given that $a > b$, which of the shaded regions is larger and why?



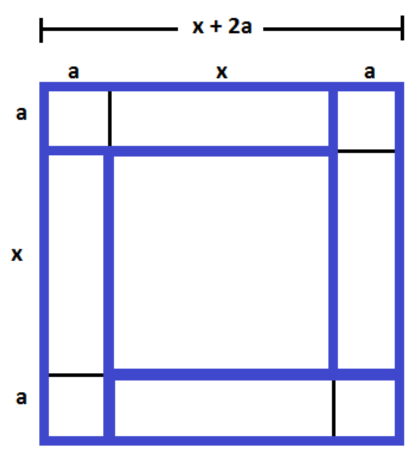
b. Consider the expressions 851×29 and 849×31. Which would result in a larger product? Use a diagram to demonstrate your result

6. Consider the following diagram.



Edna looked at the diagram and then highlighted the four small rectangles shown and concluded:

$(x + 2a)^2 = x^2 + 4a(x + a)$.



a. Michael, when he saw the picture, highlighted four rectangles and concluded:
$(x + 2a)^2 = x^2 + 2ax + 2a(x + 2a)$.
Which four rectangles and one square did he highlight?

b. Jill, when she saw the picture, highlighted eight rectangles and squares (not including the square in the middle) to conclude:
$(x + 2a)^2 = x^2 + 4ax + 4a^2$.
Which eight rectangles and squares did she highlight?

c. When Fatima saw the picture, she exclaimed: $(x + 2a)^2 = x^2 + 4a(x + 2a) - 4a^2$. She claims she highlighted just four rectangles to conclude this. Identify the four rectangles she highlighted and explain how using them she arrived that the expression $x^2 + 4a(x + 2a) - 4a^2$.

d. Is each student's technique correct? Explain why or why not.

Lesson 6: Algebraic Expressions—The Distributive Property

Problem Set Sample Solutions

1. **Insert parentheses to make each statement true.**

   a. $2 + 3 \times 4^2 + 1 = 81$ $\quad$ $(2 + 3) \times 4^2 + 1 = 81$

   b. $2 + 3 \times 4^2 + 1 = 85$ $\quad$ $(2 + 3) \times (4^2 + 1) = 85$

   c. $2 + 3 \times 4^2 + 1 = 51$ $\quad$ $2 + (3 \times 4^2) + 1 = 51$ *(or no parentheses at all is acceptable as well)*

   d. $2 + 3 \times 4^2 + 1 = 53$ $\quad$ $2 + 3 \times (4^2 + 1) = 53$

2. **Using starting symbols of w, q, 2, and -2, which of the following expressions will NOT appear when following the rules of the game played in Exercise 3?**

   a. $7w + 3q + (-2)$

   b. $q - 2$

   c. $w - q$

   d. $2w + 6$

   e. $-2w + 2$

   Expressions (c) and (e) cannot be obtained in this exercise.

   Part (d) appears as $w + w + 2 + 2 + 2$, which is equivalent to $2w + 6$.

3. **Luke wants to play the 4-number game with the numbers 1, 2, 3, and 4 and the operations of addition, multiplication, AND subtraction.**

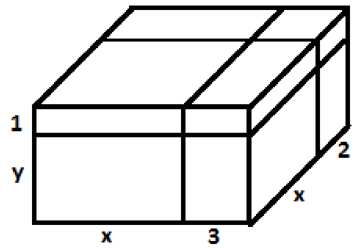
   Leoni responds, "Or we just could play the 4-number game with just the operations of addition and multiplication, but now with the numbers -1, -2, -3, -4, 1, 2, 3, and 4 instead."

   What observation is Leoni trying to point out to Luke?

   Subtraction can be viewed as the addition of a negative (e.g., $x - 4 = x + (-4)$). By introducing negative integers, we need not consider subtraction as a new operation.

4. Consider the expression: $(x+3)\cdot(y+1)\cdot(x+2)$.

   a. Draw a picture to represent the expression.

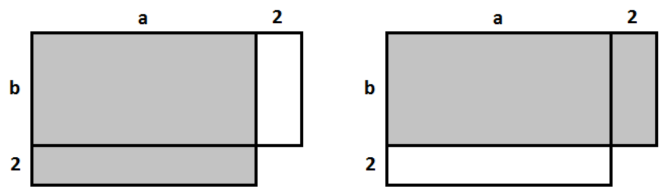


   b. Write an equivalent expression by applying the distributive property.

   $(y+1)(x+3)(x+2) = x^2y + 5xy + 6y + x^2 + 5x + 6$

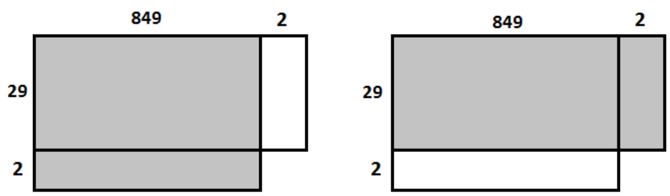
5.

   a. Given that $a > b$, which of the shaded regions is larger and why?



   The shaded region from the image on the left is larger than the shaded region from the image on the right. Both images are made up of the region of area $a \times b$ plus another region of either $2a$ (for the image on the left) or $2b$ (for the image on the right) since $a > b$, $2a > 2b$.

   b. Consider the expressions 851×29 and 849×31. Which would result in a larger product? Use a diagram to demonstrate your result.

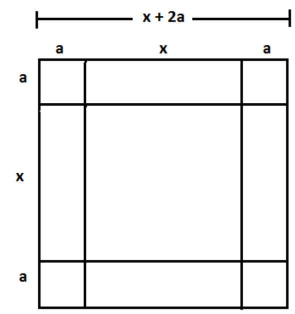


   $851 \cdot 29$ can be written as: $(849+2)29 = 849 \cdot 29 + 2 \cdot 29$ and

   $849 \cdot 31$ can be written as: $849(29+2) = 849 \cdot 29 + 2 \cdot 849$.

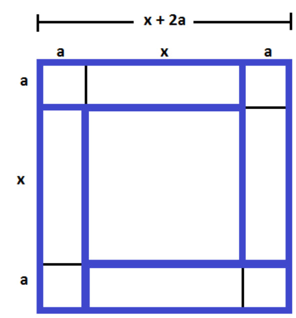
   Since $2 \cdot 29 < 2 \cdot 849$, the product $849 \cdot 31$ is the larger product.

6. Consider the following diagram.



Edna looked at the diagram and then highlighted the four small rectangles shown and concluded:

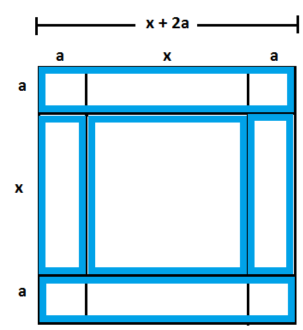
$(x+2a)^2 = x^2 + 4a(x+a)$.



a. Michael, when he saw the picture, highlighted four rectangles and concluded:

$(x+2a)^2 = x^2 + 2ax + 2a(x+2a)$.

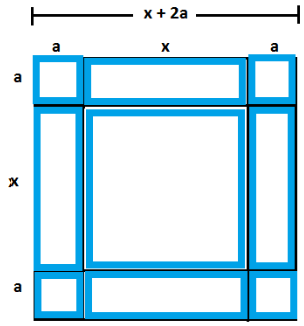
Which four rectangles and one square did he highlight?



b. **Jill, when she saw the picture, highlighted eight rectangles and squares (not including the square in the middle) to conclude:**

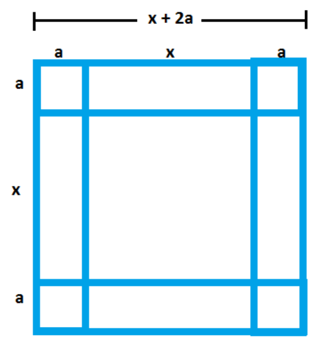
$(x + 2a)^2 = x^2 + 4ax + 4a^2$.

Which eight rectangles and squares did she highlight?



c. **When Fatima saw the picture, she exclaimed: $(x + 2a)^2 = x^2 + 4a(x + 2a) - 4a^2$. She claims she highlighted just four rectangles to conclude this. Identify the four rectangles she highlighted and explain how using them she arrived that the expression $x^2 + 4a(x + 2a) - 4a^2$.**

She highlighted each of the four rectangles that form a rim around the inner square. In doing so, she double counted each of the four $a \times a$ corners and, therefore, needed to subtract $4a^2$.



d. **Is each student's technique correct? Explain why or why not.**

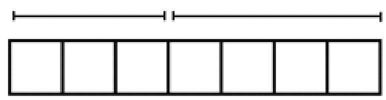
Yes, all of the techniques are right. You can see how each one is correct using the diagrams. The students broke the overall area into parts and added up the parts. In Fatima's case, she ended up counting certain areas twice and had to compensate by subtracting those areas back out of her sum.

Lesson 7: Algebraic Expressions—The Commutative and Associative Properties

Classwork

Exercise 1

Suzy draws the following picture to represent the sum $3 + 4$:

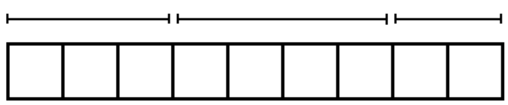


Ben looks at this picture from the opposite side of the table and says, "You drew $4 + 3$."

Explain why Ben might interpret the picture this way.

Exercise 2

Suzy adds more to her picture and says, "The picture now represents $(3 + 4) + 2$."



How might Ben interpret this picture? Explain your reasoning.

Exercise 3

Suzy then draws another picture of squares to represent the product 3×4. Ben moves to the end of the table and says, "From my new seat, your picture looks like the product 4×3."

What picture might Suzy have drawn? Why would Ben see it differently from his viewpoint?

Exercise 4

Draw a picture to represent the quantity $(3 \times 4) \times 5$ that also could represent the quantity $(4 \times 5) \times 3$ when seen from a different viewpoint.

Four Properties of Arithmetic:

<u>The Commutative Property of Addition</u>: If a and b are real numbers, then $a + b = b + a$.

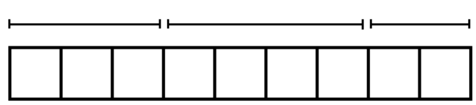
<u>The Associative Property of Addition</u>: If a, b, and c are real numbers, then $(a + b) + c = a + (b + c)$.

<u>The Commutative Property of Multiplication</u>: If a and b are real numbers, then $a \times b = b \times a$.

<u>The Associative Property of Multiplication</u>: If a, b, and c are real numbers, then $(ab)c = a(bc)$.

Exercise 5

Viewing the diagram below from two different perspectives illustrates that $(3 + 4) + 2$ equals $2 + (4 + 3)$.



Is it true for all real numbers x, y, and z that $(x + y) + z$ should equal $(z + y) + x$?

(Note: The direct application of the associative property of addition only gives $(x + y) + z = x + (y + z)$.)

Exercise 6

Draw a flow diagram and use it to prove that $(xy)z = (zy)x$ for all real numbers x, y, and z.

Exercise 7

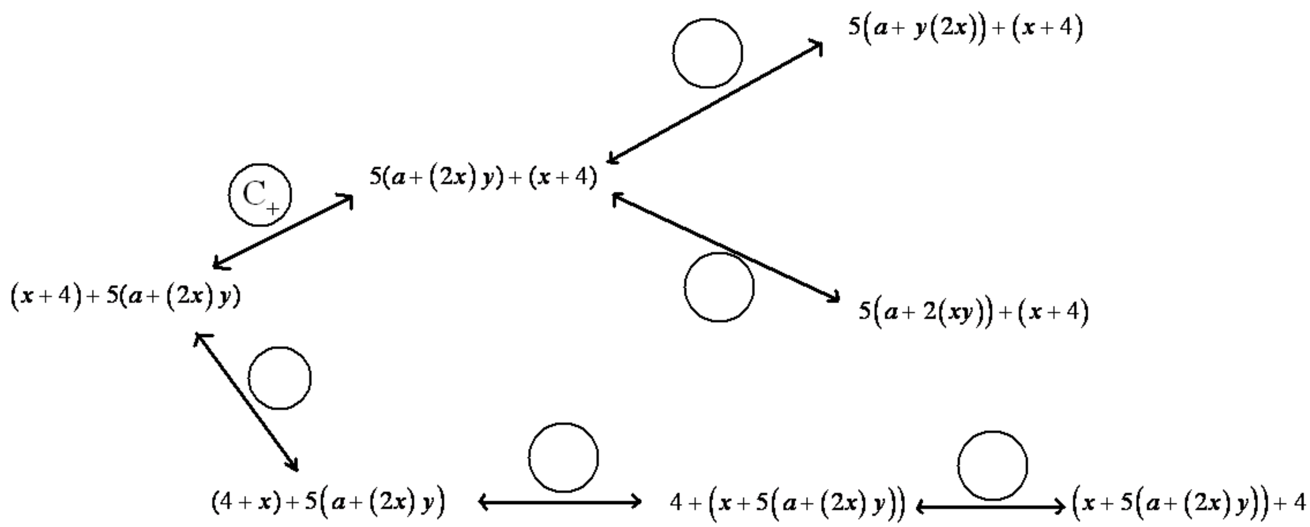
Use these abbreviations for the properties of real numbers and complete the flow diagram.

C_+ for the commutative property of addition

$C_\times$ for the commutative property of multiplication

A_+ for the associative property of addition

$A_\times$ for the associative property of multiplication



Exercise 8

Let a, b, c, and d be real numbers. Fill in the missing term of the following diagram to show that $\big((a+b)+c\big)+d$ is sure to equal $a+\big(b+(c+d)\big)$.

$$\big((a+b)+c\big)+d \xleftrightarrow{\text{A}} \big(a+(b+c)\big)+d \xleftrightarrow{\text{A}} \boxed{} \xleftrightarrow{\text{A}} a+\big(b+(c+d)\big)$$

<u>Numerical Symbol</u>: A *numerical symbol* is a symbol that represents a specific number.

For example, $0, 1, 2, 3, \frac{2}{3}, -3, -124.122, \pi, e$ are numerical symbols used to represent specific points on the real number line.

<u>Variable Symbol</u>: A *variable symbol* is a symbol that is a placeholder for a number.

It is possible that a question may restrict the type of number that a placeholder might permit; e.g., integers only or positive real numbers.

<u>Algebraic Expression</u>: An *algebraic expression* is either

1. A numerical symbol or a variable symbol, or
2. The result of placing previously generated algebraic expressions into the two blanks of one of the four operators $((\_\_)+(\_\_),\ (\_\_)-(\_\_),\ (\_\_)\times(\_\_),\ (\_\_)\div(\_\_))$ or into the base blank of an exponentiation with an exponent that is a rational number.

Two algebraic expressions are *equivalent* if we can convert one expression into the other by repeatedly applying the commutative, associative, and distributive properties and the properties of rational exponents to components of the first expression.

<u>Numerical Expression</u>: A *numerical expression* is an algebraic expression that contains only numerical symbols (no variable symbols), which evaluate to a single number.

The expression, $3 \div 0$, is not a numerical expression.

<u>Equivalent Numerical Expressions</u>: Two numerical expressions are *equivalent* if they evaluate to the same number.

Note that $1+2+3$ and $1\times 2\times 3$, for example, are equivalent numerical expressions (they are both 6) but $a+b+c$ and $a\times b\times c$ are not equivalent expressions.

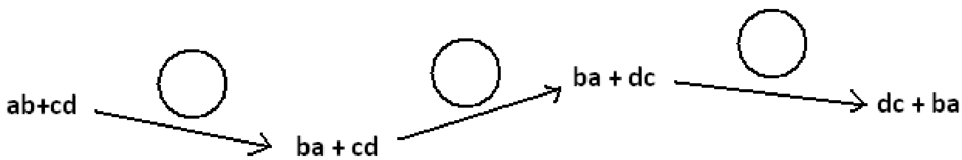
Lesson Summary

The commutative and associative properties represent key beliefs about the arithmetic of real numbers. These properties can be applied to algebraic expressions using variables that represent real numbers.

Two algebraic expressions are ***equivalent*** if we can convert one expression into the other by repeatedly applying the commutative, associative, and distributive properties and the properties of rational exponents to components of the first expression.

Problem Set

1. The following portion of a flow diagram shows that the expression $ab + cd$ is equivalent to the expression $dc + ba$.



   Fill in each circle with the appropriate symbol: Either C_+ (for the "commutative property of addition") or $C_\times$ (for the "commutative property of multiplication").

2. Fill in the blanks of this proof showing that $(w + 5)(w + 2)$ is equivalent $w^2 + 7w + 10$. Write either "commutative property," "associative property," or "distributive property" in each blank.

$(w+5)(w+2) = (w+5)w + (w+5) \times 2$ ______________________________

$= w(w+5) + (w+5) \times 2$ ______________________________

$= w(w+5) + 2(w+5)$ ______________________________

$= w^2 + w \times 5 + 2(w+5)$ ______________________________

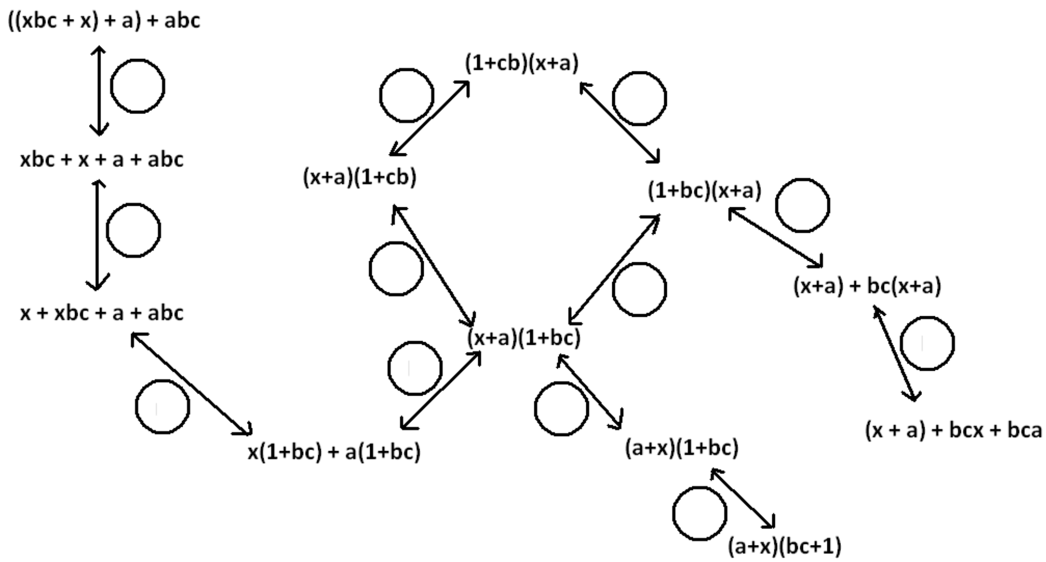
$= w^2 + 5w + 2(w+5)$ ______________________________

$= w^2 + 5w + 2w + 10$ ______________________________

$= w^2 + (5w + 2w) + 10$ ______________________________

$= w^2 + 7w + 10$

3. Fill in each circle of the following flow diagram with one of the letters: C for commutative property (for either addition or multiplication), A for associative property (for either addition or multiplication), or D for distributive property.



4. What is a quick way to see that the value of the sum $53 + 18 + 47 + 82$ is 200?

5.

   a. If $ab = 37$ and $y = \frac{1}{37}$, what is the value of the product $x \times b \times y \times a$?

   b. Give some indication as to how you used the commutative and associative properties of multiplication to evaluate $x \times b \times y \times a$ in part (a).

   c. Did you use the associative and commutative properties of addition to answer Question 4?

6. The following is a proof of the algebraic equivalency of $(2x)^3$ and $8x^3$. Fill in each of the blanks with either the statement "commutative property" or "associative property."

$(2x)^3$ $= 2x \cdot 2x \cdot 2x$

$= 2(x \times 2)(x \times 2)x$ ____________________

$= 2(2x)(2x)x$ ____________________

$= 2 \cdot 2(x \times 2)x \cdot x$ ____________________

$= 2 \cdot 2(2x)x \cdot x$ ____________________

$= (2 \cdot 2 \cdot 2)(x \cdot x \cdot x)$ ____________________

$= 8x^3$

Name ____ Date____________________

Lesson 7: Algebraic Expressions—The Commutative and Associative Properties

Exit Ticket

Write a mathematical proof of the algebraic equivalence of $(pq)r$ and $(qr)p$.

Lesson 10: True and False Equations

Classwork

Exercise 1

a. Consider the statement: "The President of the United States is a United States citizen."

Is the statement a grammatically correct sentence?

What is the subject of the sentence? What is the verb in the sentence?

Is the sentence true?

b. Consider the statement: "The President of France is a United States citizen."

Is the statement a grammatically correct sentence?

What is the subject of the sentence? What is the verb in the sentence?

Is the sentence true?

c. Consider the statement: "$2 + 3 = 1 + 4$."

This is a sentence. What is the verb of the sentence? What is the subject of the sentence?

Is the sentence true?

d. Consider the statement: "$2 + 3 = 9 + 4$."

Is this statement a sentence? And if so, is the sentence true or false?

A **number sentence** is a statement of equality between two numerical expressions.

A *number sentence* is said to be *true* if both numerical expressions are equivalent (that is, both evaluate to the same number). It is said to be *false* otherwise. True and false are called *truth values*.

Exercise 2

Determine whether the following number sentences are TRUE or FALSE.

a. $4 + 8 = 10 + 5$

b. $\frac{1}{2} + \frac{5}{8} = 1.2 - 0.075$

c. $(71 \cdot 603) \cdot 5876 = 603 \cdot (5876 \cdot 71)$

d. $13 \times 175 = 13 \times 90 + 85 \times 13$

e. $(7 + 9)^2 = 7^2 + 9^2$

f. $\pi = 3.141$

g. $\sqrt{(4 + 9)} = \sqrt{4} + \sqrt{9}$

h. $\frac{1}{2} + \frac{1}{3} = \frac{2}{5}$

i. $\frac{1}{2} + \frac{1}{3} = \frac{2}{6}$

j. $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$

k. $3^2 + 4^2 = 7^2$

l. $3^2 \times 4^2 = 12^2$

m. $3^2 \times 4^3 = 12^6$

n. $3^2 \times 3^3 = 3^5$

Exercise 3

a. Could a number sentence be both TRUE and FALSE?

b. Could a number sentence be neither TRUE nor FALSE?

An **algebraic equation** is a statement of equality between two expressions.

Algebraic equations can be number sentences (when both expressions are numerical), but often they contain symbols whose values have not been determined.

Exercise 4

a. Which of the following are algebraic equations?

i. $3.1x - 11.2 = 2.5x + 2.3$

ii. $10\pi^4 + 3 = 99\pi^2$

iii. $\pi + \pi = 2\pi$

iv. $\frac{1}{2} + \frac{1}{2} = \frac{2}{4}$

v. $79\pi^3 + 70\pi^2 - 56\pi + 87 = \frac{60\pi + 29{,}928}{\pi^2}$

b. Which of them are also number sentences?

c. For each number sentence, state whether the number sentence is true or false.

Exercises 5

When algebraic equations contain a symbol whose value has not yet been determined, we use analysis to determine whether:

1. The equation is true for all the possible values of the variable(s), or
2. The equation is true for a certain set of the possible value(s) of the variable(s), or
3. The equation is never true for any of the possible values of the variable(s).

For each of the three cases, write an algebraic equation that would be correctly described by that case. Use only the variable, x, where x represents a real number.

Example 1

Consider the following scenario.

Julie is 300 feet away from her friend's front porch and observes, "Someone is sitting on the porch."

Given that she didn't specify otherwise, we would assume that the "someone" Julie thinks she sees is a human. We can't guarantee that Julie's observatory statement is true. It could be that Julie's friend has something on the porch that merely looks like a human from far away. Julie assumes she is correct and moves closer to see if she can figure out who it is. As she nears the porch she declares, "Ah, it is our friend, John Berry."

Exercise 6

Name a value of the variable that would make each equation a true number sentence.

Here are several examples of how we can name the value of the variable:

Let $w = -2$. Then $w^2 = 4$ is true.

or

$w^2 = 4$ is true when $w = -2$

or

$w^2 = 4$ is true if $w = -2$

or

$w^2 = 4$ is true for $w = -2$ and $w = 2$.

There might be more than one option for what numerical values to write. (And feel free to write more than one possibility.)

Warning: Some of these are tricky. Keep your wits about you!

a. Let ____________________. Then $7 + x = 12$ is true.

b. Let ____________________. Then $3r + 0.5 = \frac{37}{2}$ is true.

c. $m^3 = -125$ is true for ______________________.

d. A number x and its square, x^2, have the same value when ______________________.

e. The average of 7 and n is -8 if ______________________.

f. Let _________________________. Then $2a = a + a$ is true.

g. $q + 67 = q + 68$ is true for ______________________.

Problem Set

Determine whether the following number sentences are true or false.

1. $18 + 7 = \frac{50}{2}$
2. $3.123 = 9.369 \cdot \frac{1}{3}$
3. $(123 + 54) \cdot 4 = 123 + (54 \cdot 4)$
4. $5^2 + 12^2 = 13^2$
5. $(2 \times 2)^2 = \sqrt{256}$
6. $\frac{4}{3} = 1.333$

In the following equations, let $x = -3$ and $y = \frac{2}{3}$. Determine whether the following equations are true, false, or neither true nor false.

7. $xy = -2$
8. $x + 3y = -1$
9. $x + z = 4$
10. $9y = -2x$
11. $\frac{y}{x} = -2$
12. $\frac{-\frac{2}{x}}{y} = -1$

For each of the following, assign a value to the variable, x, to make the equation a true statement.

13. $(x^2 + 5)(3 + x^4)(100x^2 - 10)(100x^2 + 10) = 0$ for ____________________.
14. $\sqrt{(x+1)(x+2)} = \sqrt{20}$ for ____________________.
15. $(d + 5)^2 = 36$ for ____________________.
16. $(2z + 2)(z^5 - 3) + 6 = 0$ for ____________________.
17. $\frac{1+x}{1+x^2} = \frac{3}{5}$ for ____________________.
18. $\frac{1+x}{1+x^2} = \frac{2}{5}$ for ____________________.
19. The diagonal of a square of side length L is 2 inches long when ____________________.
20. $\left(T - \sqrt{3}\right)^2 = T^2 + 3$ for ____________________.
21. $\frac{1}{x} = \frac{x}{1}$ if ____________________.

22. $\left(2+\left(2-\left(2+\left(2-(2+r)\right)\right)\right)\right)=1$ for ____________________.

23. $x+2=9$

24. $x+2^2=-9$

25. $-12t=12$

26. $12t=24$

27. $\frac{1}{b-2}=\frac{1}{4}$

28. $\frac{1}{2b-2}=-\frac{1}{4}$

29. $\sqrt{x}+\sqrt{5}=\sqrt{x+5}$

30. $(x-3)^2=x^2+(-3)^2$

31. $x^2=-49$

32. $\frac{2}{3}+\frac{1}{5}=\frac{3}{x}$

Fill in the blank with a variable term so that the given value of the variable will make the equation true.

33. _____ $+4=12;\ x=8$

34. _____ $+4=12;\ x=4$

Fill in the blank with a constant term so that the given value of the variable will make the equation true.

35. $4y-$ ____ $=100;\ y=25$

36. $4y-$ ____ $=0;\ y=6$

37. $r+$ ____ $=r;$ r is any real number

38. $r\times$ ____ $=r;$ r is any real number

Generate the following:

39. An equation that is always true

40. An equation that is true when $x=0$

41. An equation that is never true

42. An equation that is true when $t=1$ or $t=-1$

43. An equation that is true when $y=-0.5$

44. An equation that is true when $z=\pi$

Lesson 1: Integer Sequences—Should You Believe in Patterns?

Classwork

Opening Exercise

Mrs. Rosenblatt gave her students what she thought was a very simple task:

What is the next number in the sequence $2, 4, 6, 8, \ldots$?

Cody: I am thinking of a "plus 2 *pattern," so it continues* $10, 12, 14, 16, \ldots$.

Ali: I am thinking a repeating pattern, so it continues $2, 4, 6, 8, 2, 4, 6, 8, \ldots$.

Suri: I am thinking of the units digits in the multiples of two, so it continues $2, 4, 6, 8, 0, 2, 4, 6, 8, \ldots$.

1. Are each of these valid responses?

2. What is the hundredth number in the sequence in Cody's scenario? Ali's? Suri's?

3. What is an expression in terms of n for the n^{th} number in the sequence in Cody's scenario?

Example 1

Jerry has thought of a pattern that shows powers of two. Here are the first 6 numbers of Jerry's sequence:

$1, 2, 4, 8, 16, 32, \ldots$

Write an expression for the n^{th} number of Jerry's sequence.

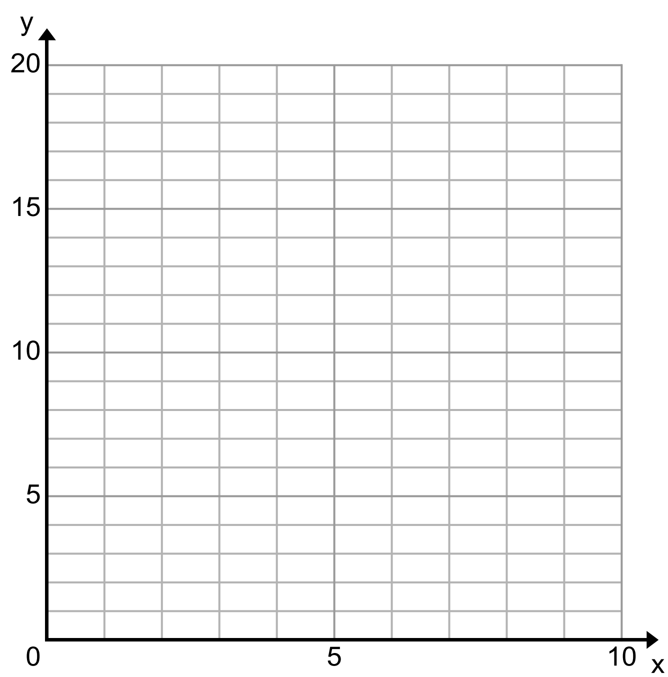
Example 2

Consider the sequence that follows a “plus 3” pattern: $4, 7, 10, 13, 16, \ldots$.

a. Write a formula for the sequence using both the a_n notation and the $f(n)$ notation.

b. Does the formula $f(n) = 3(n - 1) + 4$ generate the same sequence? Why might some people prefer this formula?

c. Graph the terms of the sequence as ordered pairs $(n, f(n))$ on the coordinate plane. What do you notice about the graph?



Exercises

1. Refer back to the sequence from the Opening Exercise. When Ms. Rosenblatt was asked for the next number in the sequence $2, 4, 6, 8, \ldots$, she said 17. The class responded, "17?"

   Yes, using the formula, $f(n) = \frac{7}{24}(n-1)^4 - \frac{7}{4}(n-1)^3 + \frac{77}{24}(n-1)^2 + \frac{1}{4}(n-1) + 2$.

   a. Does her formula actually produce the numbers 2, 4, 6, and 8?

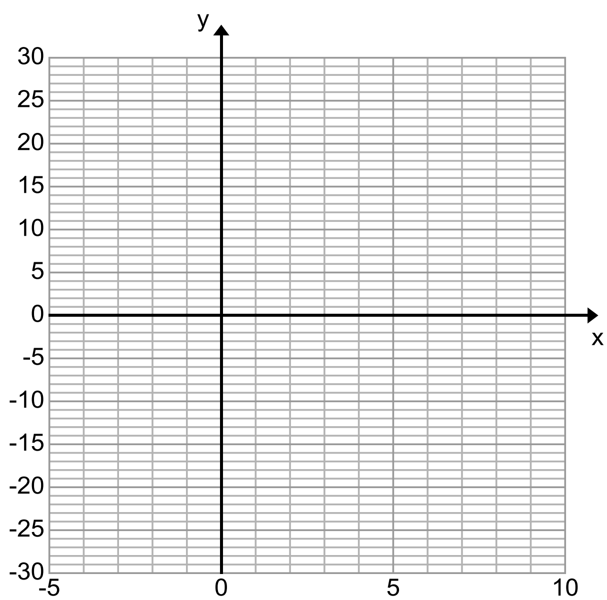
   b. What is the 100^{th} term in Ms. Rosenblatt's sequence?

2. Consider a sequence that follows a "minus 5" pattern: $30, 25, 20, 15, \ldots$.

   a. Write a formula for the nth term of the sequence. Be sure to specify what value of n your formula starts with.

   b. Using the formula, find the 20^{th} term of the sequence.

   c. Graph the terms of the sequence as ordered pairs $(n, f(n))$ on a coordinate plane.

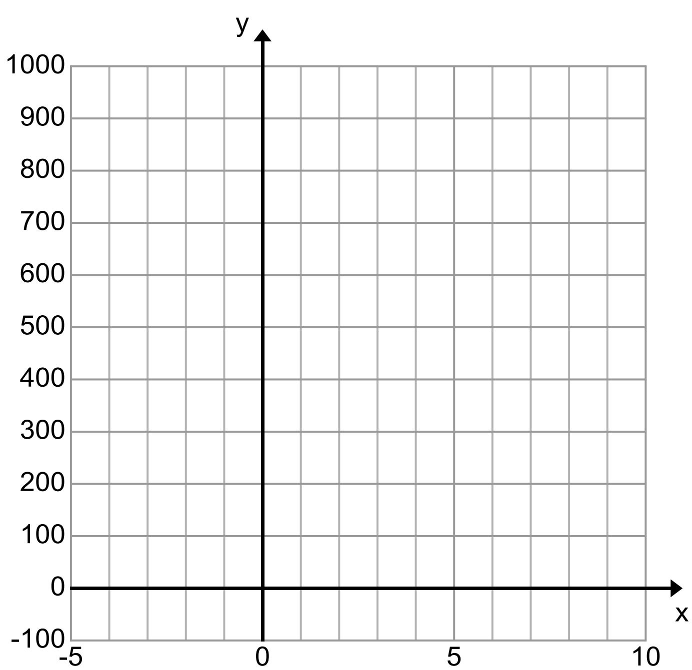


3. Consider a sequence that follows a "times 5" pattern: $1, 5, 25, 125, \ldots$.

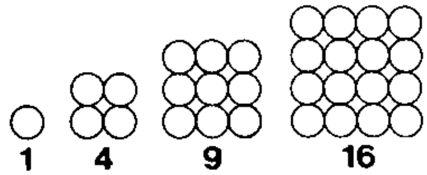
   a. Write a formula for the n^{th} term of the sequence. Be sure to specify what value of n your formula starts with.

   b. Using the formula, find the 10^{th} term of the sequence.

   c. Graph the terms of the sequence as ordered pairs $(n, f(n))$ on a coordinate plane.



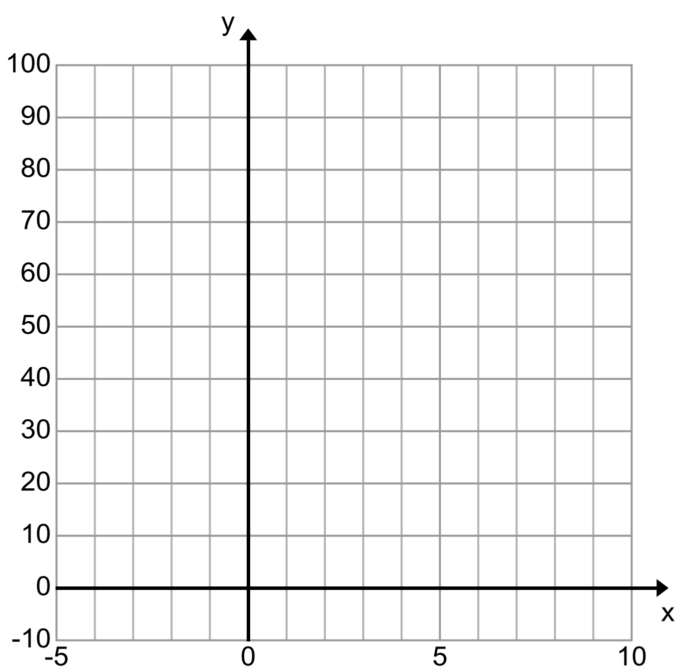
4. Consider the sequence formed by the square numbers:



a. Write a formula for the nth term of the sequence. Be sure to specify what value of n your formula starts with.

b. Using the formula, find the 50^{th} term of the sequence.

c. Graph the terms of the sequence as ordered pairs $\left(n, f(n)\right)$ on a coordinate plane.



5. A standard letter-sized piece of paper has a length and width of 8.5 inches by 11 inches.

   a. Find the area of one piece of paper.

   b. If the paper were folded completely in half, what would be the area of the resulting rectangle?

   c. Write a formula for a sequence to determine the area of the paper after n folds.

   d. What would the area be after 7 folds?

Lesson Summary

A sequence can be thought of as an ordered list of elements. To define the pattern of the sequence, an explicit formula is often given, and unless specified otherwise, the first term is found by substituting 1 into the formula.

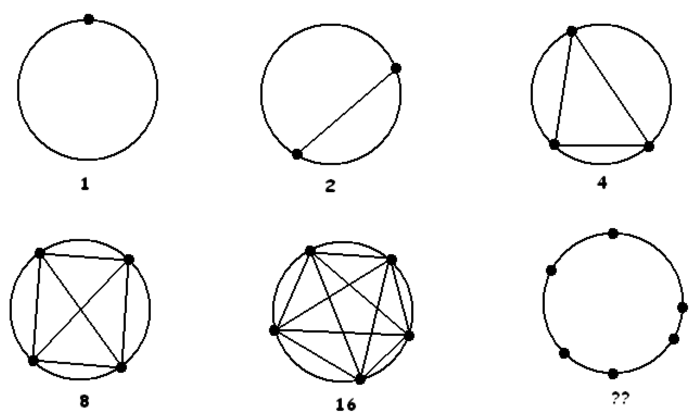
Problem Set

1. Consider a sequence generated by the formula $f(n) = 6n - 4$ starting with $n = 1$. Generate the terms $f(1)$, $f(2)$, $f(3)$, $f(4)$, and $f(5)$.

2. Consider a sequence given by the formula $f(n) = \frac{1}{3^{n-1}}$ starting with $n = 1$. Generate the first 5 terms of the sequence.

3. Consider a sequence given by the formula $f(n) = (-1)^n \times 3$ starting with $n = 1$. Generate the first 5 terms of the sequence.

4. Here is the classic puzzle that shows that patterns need not hold true. What are the numbers counting?



   a. Based on the sequence of numbers, predict the next number.
   b. Write a formula based on the perceived pattern.
   c. Find the next number in the sequence by actually counting.
   d. Based on your answer from part (c), is your model from part (b) effective for this puzzle?

In Problems 5-8, for each of the following sequences:

a. Write a formula for the n^{th} term of the sequence. Be sure to specify what value of n your formula starts with.

b. Using the formula, find the 15^{th} term of the sequence.

c. Graph the terms of the sequence as ordered pairs $(n, f(n))$ on a coordinate plane.

5. The sequence follows a "plus 2" pattern: $3, 5, 7, 9, \ldots$.

6. The sequence follows a "times 4" pattern: $1, 4, 16, 64, \ldots$.

7. The sequence follows a "times -1" pattern: $6, -6, 6, -6, \ldots$.

8. The sequence follows a "minus 3" pattern: $12, 9, 6, 3, \ldots$.

Lesson 5: The Power of Exponential Growth

Classwork

Opening Exercise

Two equipment rental companies have different penalty policies for returning a piece of equipment late:

Company 1: On day 1, the penalty is $\$5$. On day 2, the penalty is $\$10$. On day 3, the penalty is $\$15$. On day 4, the penalty is $\$20$ and so on, increasing by $\$5$ each day the equipment is late.

Company 2: On day 1, the penalty is $\$0.01$. On day 2, the penalty is $\$0.02$. On day 3, the penalty is $\$0.04$. On day 4, the penalty is $\$0.08$ and so on, doubling in amount each additional day late.

Jim rented a digger from Company 2 because he thought it had the better late return policy. The job he was doing with the digger took longer than he expected, but it did not concern him because the late penalty seemed so reasonable. When he returned the digger 15 days late, he was shocked by the penalty fee. What did he pay, and what would he have paid if he had used Company 1 instead?

Company 1	
Day	Penalty
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
13	
14	
15	

Company 2	
Day	Penalty
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
13	
14	
15	

1. Which company has a greater 15-day late charge?

2. Describe how the amount of the late charge changes from any given day to the next successive day in both Companies 1 and 2.

3. How much would the late charge have been after 20 days under Company 2?

Example 1

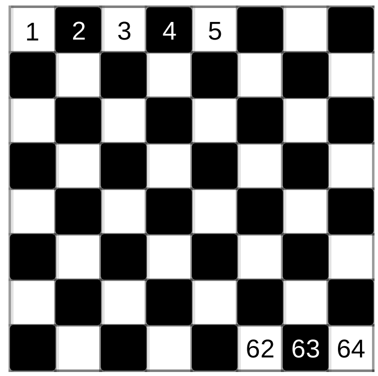
Folklore suggests that when the creator of the game of chess showed his invention to the country's ruler, the ruler was highly impressed. He was so impressed, he told the inventor to name a prize of his choice. The inventor, being rather clever, said he would take a grain of rice on the first square of the chessboard, two grains of rice on the second square of the chessboard, four on the third square, eight on the fourth square, and so on, doubling the number of grains of rice for each successive square. The ruler was surprised, even a little offended, at such a modest prize, but he ordered his treasurer to count out the rice.

a. Why is the ruler "surprised"? What makes him think the inventor requested a "modest prize"?

The treasurer took more than a week to count the rice in the ruler's store, only to notify the ruler that it would take more rice than was available in the entire kingdom. Shortly thereafter, as the story goes, the inventor became the new king.

b. Imagine the treasurer counting the needed rice for each of the 64 squares. We know that the first square is assigned a single grain of rice, and each successive square is double the number of grains of rice of the former square. The following table lists the first five assignments of grains of rice to squares on the board. How can we represent the grains of rice as exponential expressions?

Square #	Grains of Rice	Exponential Expression
1	1	
2	2	
3	4	
4	8	
5	16	



c. Write the exponential expression that describes how much rice is assigned to each of the last three squares of the board.

Square #	Exponential Expression
62	
63	
64	

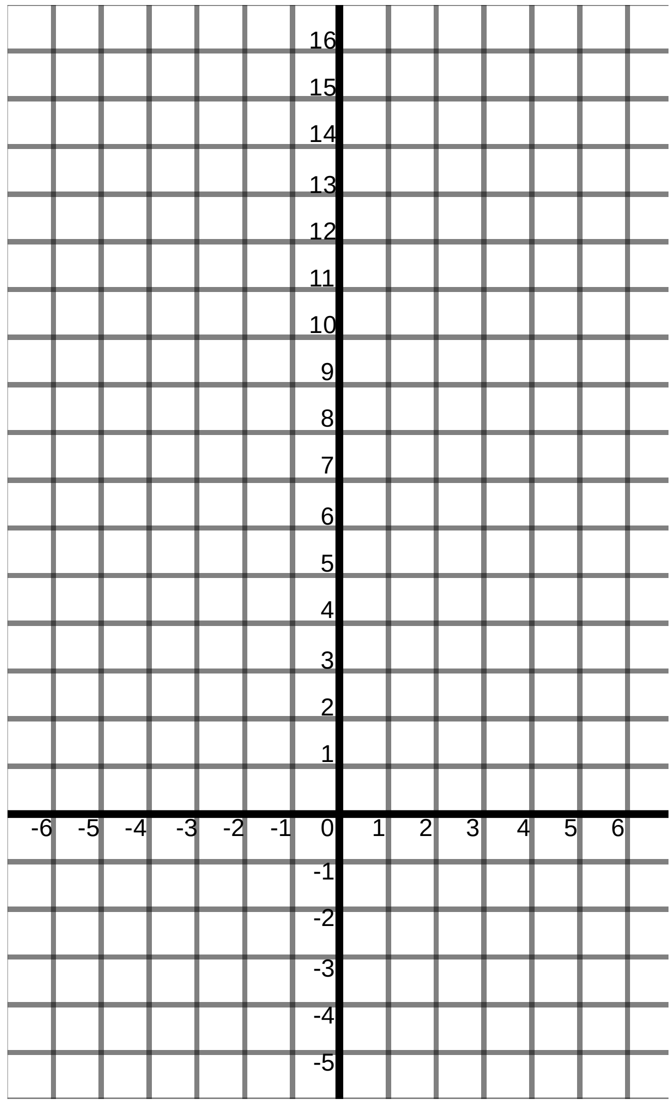
Example 2

Let us understand the difference between $f(n) = 2n$ and $f(n) = 2^n$.

a. Complete the tables below, and then graph the points $(n, f(n))$ on a coordinate plane for each of the formulas.

n	$f(n) = 2n$
-2	
-1	
0	
1	
2	
3	

n	$f(n) = 2^n$
-2	
-1	
0	
1	
2	
3	



b. Describe the change in each sequence when n increases by 1 unit for each sequence.

Exercise 1

A typical thickness of toilet paper is 0.001 inches. Seems pretty thin, right? Let's see what happens when we start folding toilet paper.

a. How thick is the stack of toilet paper after 1 fold? After 2 folds? After 5 folds?

b. Write an explicit formula for the sequence that models the thickness of the folded toilet paper after n folds.

c. After how many folds will the stack of folded toilet paper pass the 1 foot mark?

d. The moon is about $240{,}000$ miles from Earth. Compare the thickness of the toilet paper folded 50 times to the distance from Earth.

Watch the following video "How folding paper can get you to the moon"
(http://www.youtube.com/watch?v=AmFMJC45f1Q)

Exercise 2

A rare coin appreciates at a rate of 5.2% a year. If the initial value of the coin is $\$500$, after how many years will its value cross the $\$3{,}000$ mark? Show the formula that will model the value of the coin after t years.

Discussion (10 minutes): Exponential Formulas

Ask students to consider how the exponential expressions of Example 1, part (b) relate to one another.

- Why is the base of the expression 2?
  - *Since each successive square has twice the amount of rice as the former square, the factor by which the rice increases is a factor of* 2.

MP.4

- What is the explicit formula for the sequence that models the number of rice grains in each square? Use n to represent the number of the square and $f(n)$ to represent the number of rice grains assigned to that square.
  - $f(n) = 2^{(n-1)}$, *where* $f(n)$ *represents the number of rice grains belonging to each square, and* n *represents the number of the square on the board.*
- Would the formula $f(n) = 2^n$ work? Why or why not?
  - *No, the formula is supposed to model the numbering scheme on the chessboard corresponding to the story.*
- What would have to change for the formula $f(n) = 2^n$ to be appropriate?
  - *If the first square started with* 2 *grains of rice and doubled thereafter, or if we numbered the squares as starting with square number* 0 *and ending on square* 63, *then* $f(n) = 2^n$ *would be appropriate.*
- Suppose instead that the first square did not begin with a single grain of rice but with 5 grains of rice, and then the number of grains was doubled with each successive square. Write the sequence of numbers representing the number of grains of rice for the first five squares.
  - $5, 10, 20, 40, 80$
- Suppose we wanted to represent these numbers using exponents? Would we still require the use of the powers of 2?
  - *Yes.* $5 = 5(2^0), 10 = 5(2^1), 20 = 5(2^2), 40 = 5(2^3), 80 = 5(2^4)$
- Generalize the pattern of these exponential expressions into an explicit formula for the sequence. How does it compare to the formula in the case where we began with a single grain of rice in the first square?
  - $f(n) = 5(2^{n-1})$, *the powers of* 2 *cause the doubling effect, and the* 5 *represents the initial* 5 *grains of rice.*
- Generalize the formula even further. Write a formula for a sequence that allows for any possible value for the number of grains of rice on the first square.
  - $f(n) = a2^{n-1}$, *where* a *represents the number of rice grains on the first square.*
- Generalize the formula even further. What if instead of doubling the number of grains, we wanted to triple or quadruple them?
  - $f(n) = ab^{n-1}$, *where* a *represents the number of rice grains on the first square and* b *represents the factor by which the number of rice grains is multiplied on each successive square.*
- Is the sequence for this formula Geometric, Arithmetic, or neither?
  - *Geometric*

Name ____ Date____________________

Lesson 5: The Power of Exponential Growth

Exit Ticket

Chain emails are emails with a message suggesting you will have good luck if you forward the email on to others. Suppose a student started a chain email by sending the message to 3 friends and asking those friends to each send the same email to 3 more friends exactly 1 day after they received it.

a. Write an explicit formula for the sequence that models the number of people who will receive the email on the n^{th} day. (Let the first day be the day the original email was sent.) Assume everyone who receives the email follows the directions.

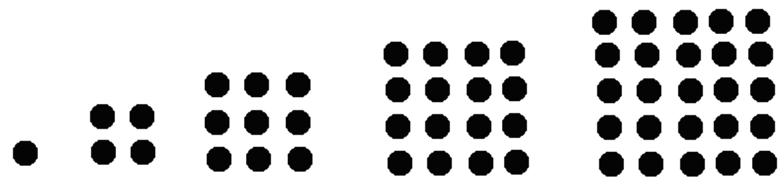
b. Which day will be the first day that the number of people receiving the email exceeds 100?

Lesson 8: Why Stay with Whole Numbers?

Classwork

Opening Exercise

The sequence of perfect squares $\{1, 4, 9, 16, 25, \dots\}$ earned its name because the ancient Greeks realized these quantities could be arranged to form square shapes.



1st **2nd** **3rd** **4th** **5th**

If $S(n)$ denotes the nth square number, what is a formula for $S(n)$?

Exercises

1. Prove whether or not 169 is a perfect square.

2. Prove whether or not 200 is a perfect square.

3. If $S(n) = 225$, then what is n?

4. Which term is the number 400 in the sequence of perfect squares? How do you know?

Instead of arranging dots into squares, suppose we extend our thinking to consider squares of side length x cm.

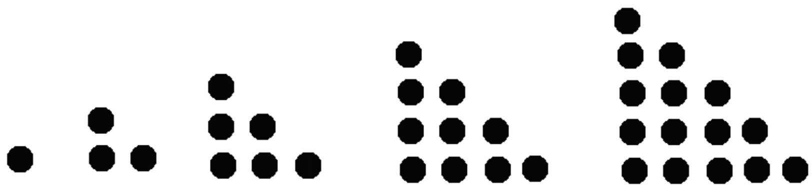
5. Create a formula for the area $A(x)$ cm^2 of a square of side length x cm. $A(x) =$ ___________

6. Use the formula to determine the area of squares with side lengths of 3 cm, 10.5 cm, and π cm.

7. What does $A(0)$ mean?

8. What does $A(-10)$ and $A(\sqrt{2})$ mean?

The triangular numbers are the numbers that arise from arranging dots into triangular figures as shown:



9. What is the 100^{th} triangular number?

10. Find a formula for $T(n)$, the nth triangular number (starting with $n = 1$).

11. How can you be sure your formula works?

12. Create a graph of the sequence of triangular numbers, $(n) = \frac{n(n+1)}{2}$, where n is a positive integer.

13. Create a graph of the triangle area formula $T(x) = \frac{x(x+1)}{2}$, where x is any positive real number.

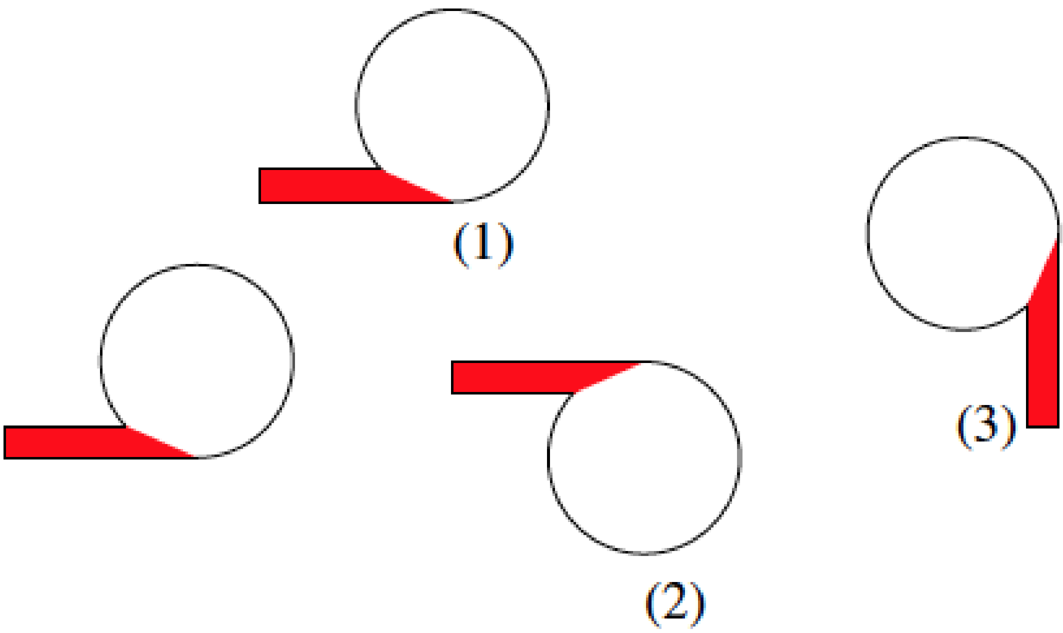
14. How are your two graphs alike? How are they different?

Lesson 1: Why Move Things Around?

Classwork

Exploratory Challenge 1

1. Describe, *intuitively*, what kind of transformation will be required to move the figure on the left to each of the figures $(1-3)$ on the right. To help with this exercise, use a transparency to copy the figure on the left. Note that you are supposed to begin by moving the left figure to each of the locations in (1), (2), and (3).



Lesson 17: Four Interesting Transformations of Functions

Classwork

Exploratory Challenge 1/Example 1

Let $f(x) = |x|$, $g(x) = f(x) - 3$, $h(x) = f(x) + 2$ for any real number x.

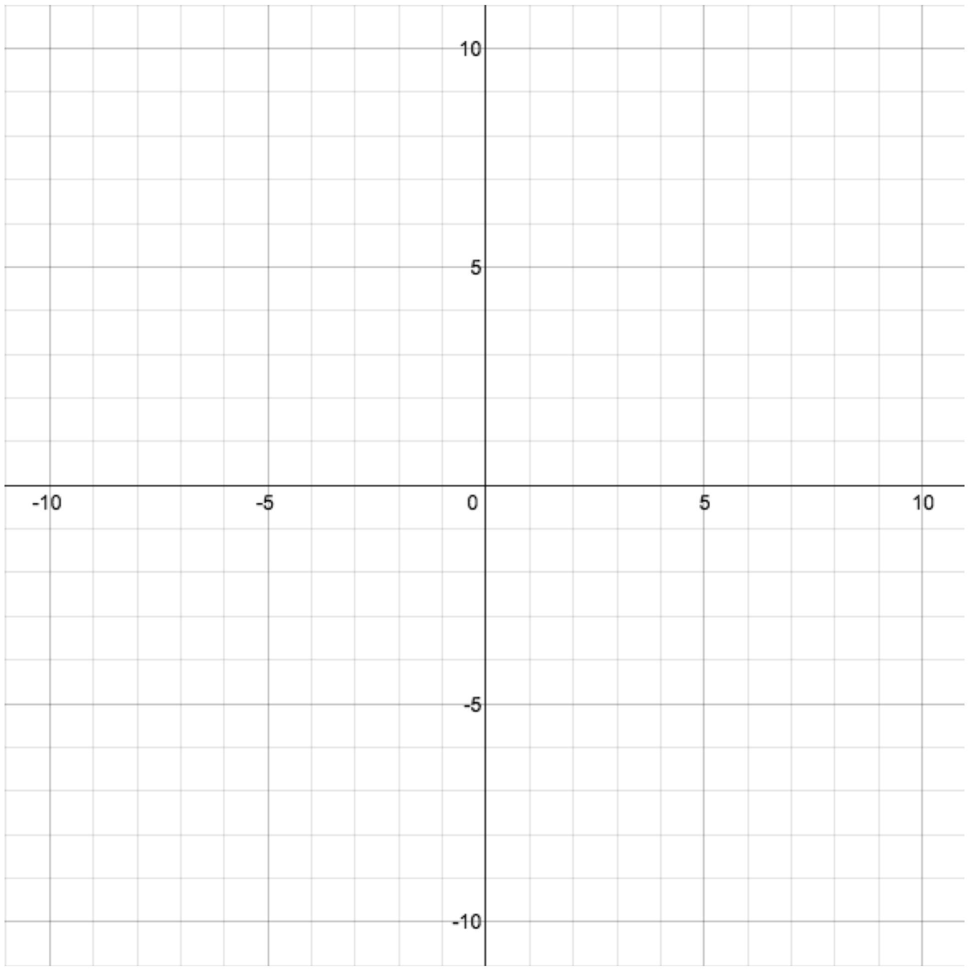
1. Write an explicit formula for $g(x)$ in terms of $|x|$ (i.e., without using $f(x)$ notation):

2. Write an explicit formula for $h(x)$ in terms of $|x|$ (i.e., without using $f(x)$ notation):

3. Complete the table of values for these functions.

x	$f(x) = \|x\|$	$g(x) = f(x) - 3$	$h(x) = f(x) + 2$
-3			
-2			
-1			
0			
1			
2			
3			

4. Graph all three equations: $y = f(x)$, $y = f(x) - 3$, and $y = f(x) + 2$.



5. What is the relationship between the graph of $y = f(x)$ and the graph of $y = f(x) + k$?

6. How do the values of g and h relate to the values of f?

Exploratory Challenge 2/Example 2

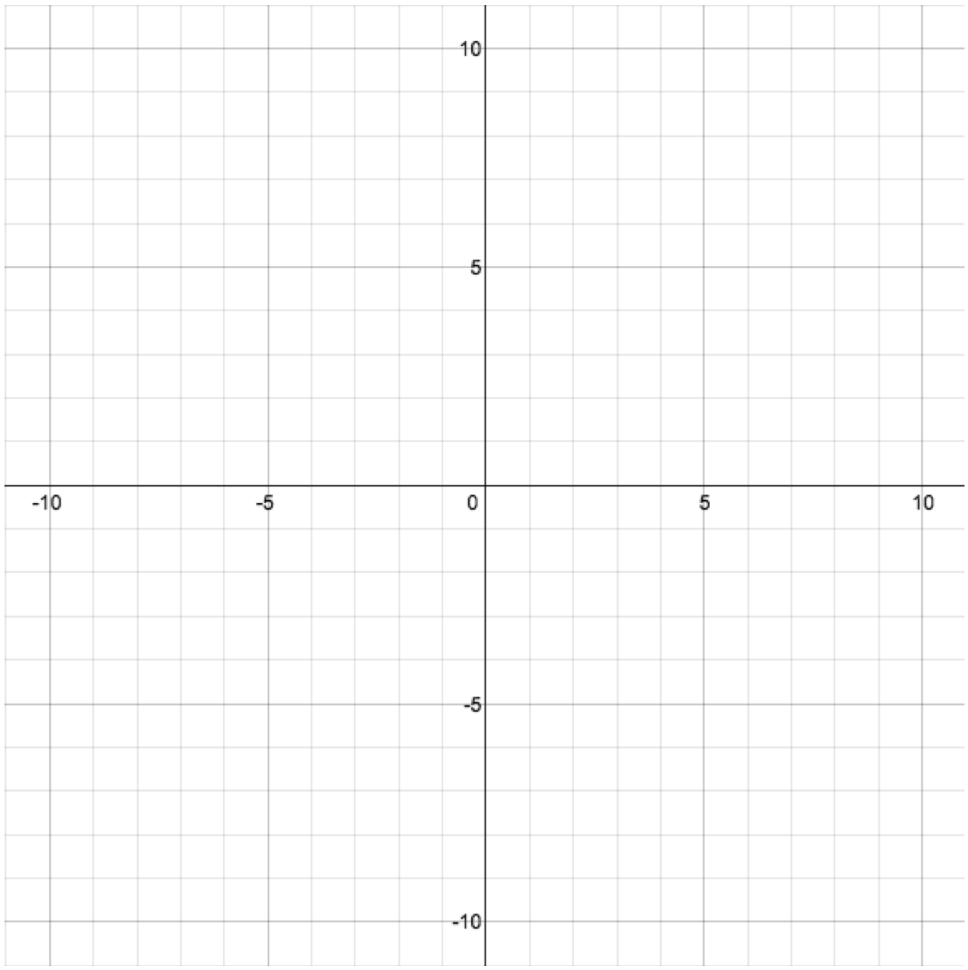
1. Let $f(x) = |x|$, $g(x) = 2f(x)$, $h(x) = \frac{1}{2}f(x)$ for any real number x. Write a formula for $g(x)$ in terms of $|x|$ (i.e., without using $f(x)$ notation):

2. Write a formula for $h(x)$ in terms of $|x|$ (i.e., without using $f(x)$ notation):

3. Complete the table of values for these functions.

x	$f(x) = \lvert x\rvert$	$g(x) = 2f(x)$	$h(x) = \frac{1}{2}f(x)$
-3			
-2			
-1			
0			
1			
2			
3			

4. Graph all three equations: $y = f(x)$, $y = 2f(x)$, and $y = \frac{1}{2}f(x)$.



Let $p(x) = -|x|$, $q(x) = -2f(x)$, $r(x) = -\frac{1}{2}f(x)$ for any real number x.

5. Write the formula for $q(x)$ in terms of $|x|$ (i.e., without using $f(x)$ notation):

6. Write the formula for $r(x)$ in terms of $|x|$ (i.e., without using $f(x)$ notation):

7. Complete the table of values for the functions $p(x) = -|x|$, $q(x) = -2f(x)$, $r(x) = -\frac{1}{2}f(x)$.

x	$p(x) = -\lvert x \rvert$	$q(x) = -2f(x)$	$r(x) = -\frac{1}{2}f(x)$
-3			
-2			
-1			
0			
1			
2			
3			

8. Graph all three functions on the same graph as $y = p(x)$, $y = q(x)$, and $y = r(x)$.

9. How is the graph of $y = f(x)$ related to the graph of $y = kf(x)$ when $k > 1$?

10. How is the graph of $y = f(x)$ related to the graph of $y = kf(x)$ when $0 < k < 1$?

11. How do the values of functions p, q, and r relate to the values of functions f, g, and h, respectively? What transformation of the graphs of f, g, and h represents this relationship?

Lesson 18: Four Interesting Transformations of Functions

Classwork

Example 1

Let $f(x) = |x|$, $g(x) = f(x-3)$, $h(x) = f(x+2)$ where x can be any real number.

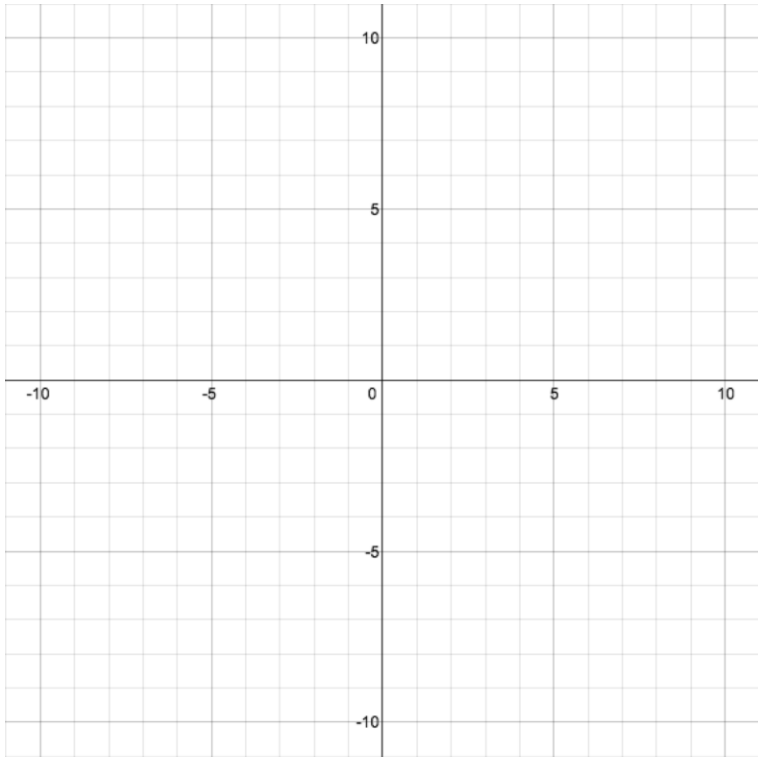
a. Write the formula for $g(x)$ in terms of $|x|$ (i.e., without using $f(x)$ notation):

b. Write the formula for $h(x)$ in terms of $|x|$ (i.e., without using $f(x)$ notation):

c. Complete the table of values for these functions.

x	$f(x) = \lvert x\rvert$	$g(x) =$	$h(x) =$
-3			
-2			
-1			
0			
1			
2			
3			

d. Graph all three equations: $y = f(x)$, $y = f(x - 3)$, and $y = f(x + 2)$.



e. How does the graph of $y = f(x)$ relate to the graph of $y = f(x - 3)$?

f. How does the graph of $y = f(x)$ relate to the graph of $y = f(x + 2)$?

g. How does the graph of $y = |x| - 3$ and the graph of $y = |x - 3|$ relate differently to the graph of $y = |x|$?

h. How do the values of g and h relate to the values of f?

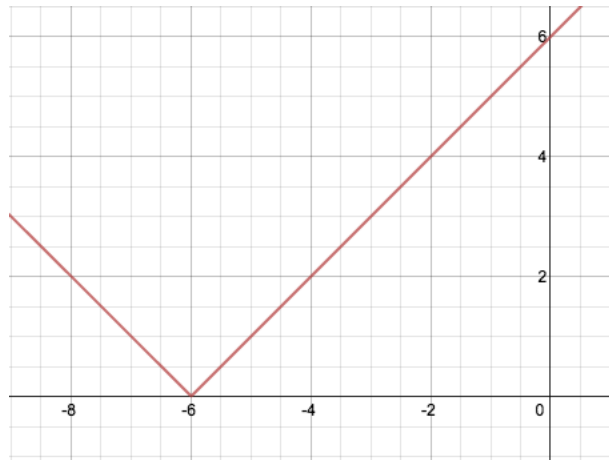
Exercises

1. Karla and Isamar are disagreeing over which way the graph of the function $g(x) = |x + 3|$ is translated relative to the graph of $f(x) = |x|$. Karla believes the graph of g is "to the right" of the graph of f; Isamar believes the graph is "to the left." Who is correct? Use the coordinates of the vertex of f and g and to support your explanation.

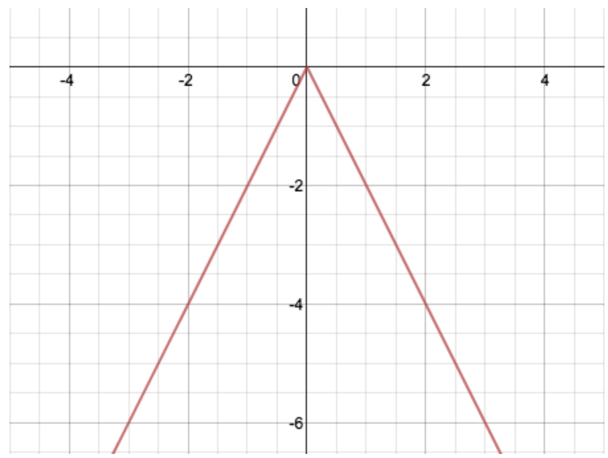
2. Let $f(x) = |x|$ where x can be any real number. Write a formula for the function whose graph is the transformation of the graph of f given by the instructions below.
   a. A translation right 5 units.
   b. A translation down 3 units.
   c. A vertical scaling (a vertical stretch) with scale factor of 5.
   d. A translation left 4 units.
   e. A vertical scaling (a vertical shrink) with scale factor of $\frac{1}{3}$.

3. Write the formula for the function depicted by the graph.

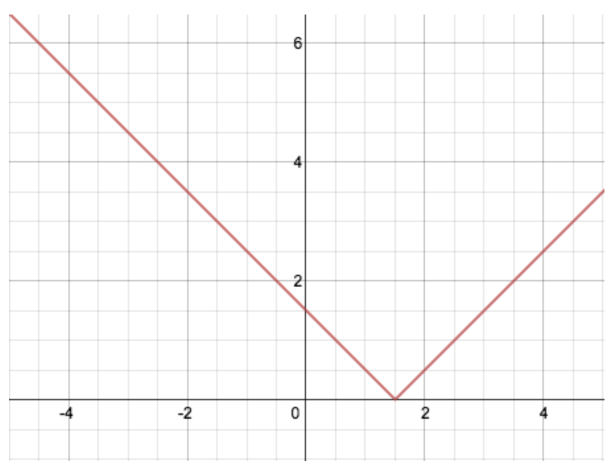
a. $y =$



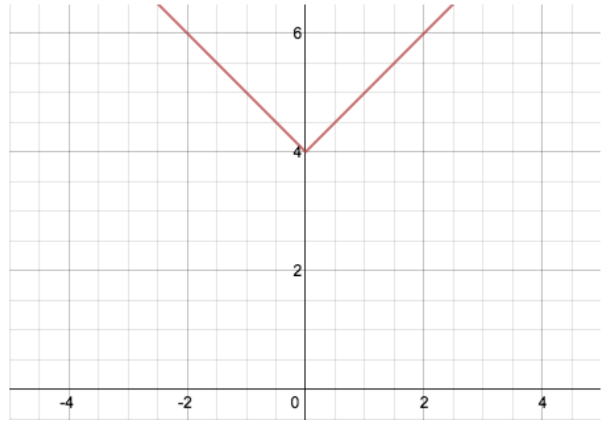
b. $y =$



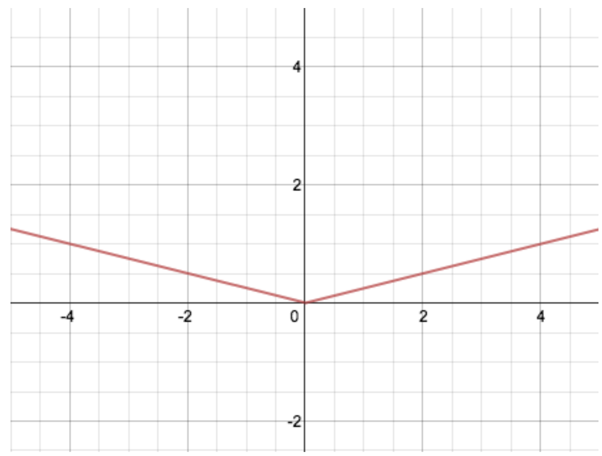
c. $y =$



d. $y =$



e. $y =$



4. Let $f(x) = |x|$ where x can be any real number. Write a formula for the function whose graph is the described transformation of the graph of f.

   a. A translation 2 units left and 4 units down.

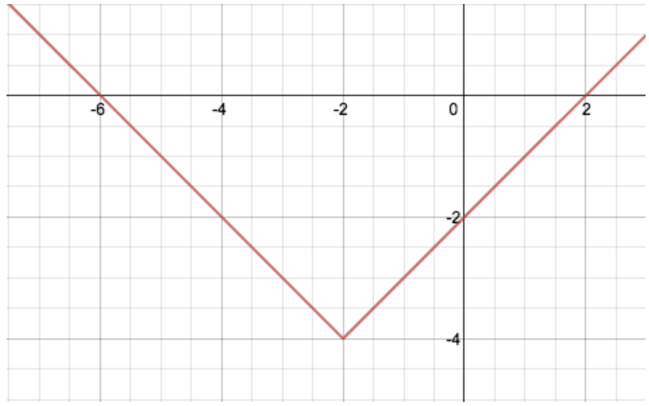
   b. A translation 2.5 units right and 1 unit up.

   c. A vertical scaling with scale factor $\frac{1}{2}$ and then a translation 3 units right.

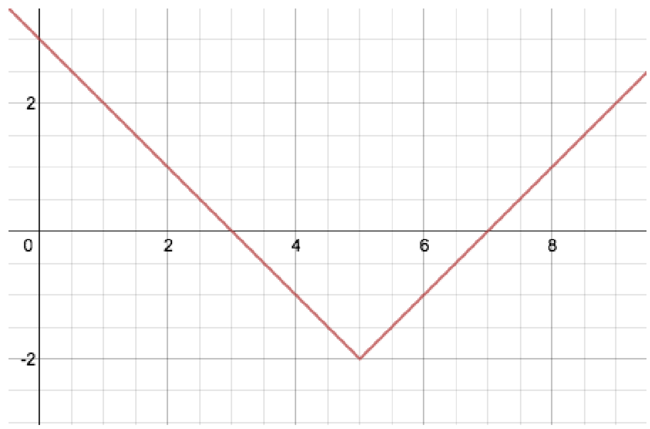
   d. A translation 5 units right and a vertical scaling by reflecting across the x-axis with vertical scale factor -2.

5. Write the formula for the function depicted by the graph.

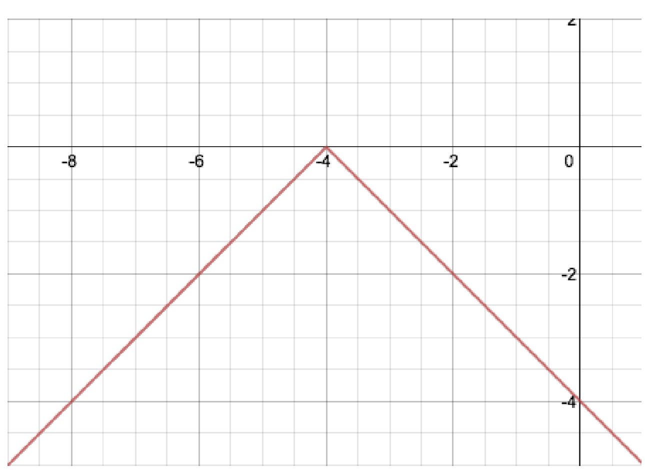
a. $y =$



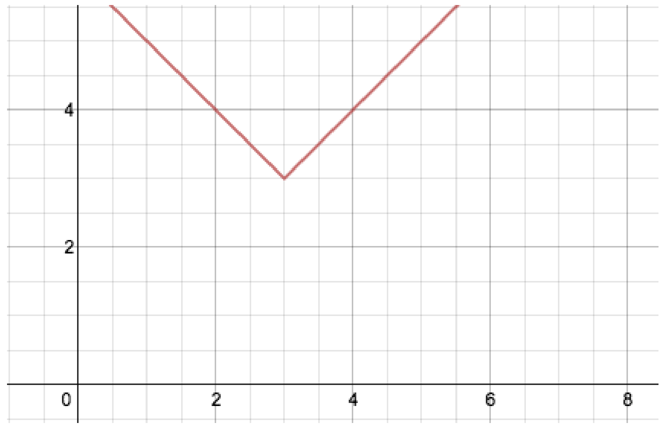
b. $y =$



c. $y =$



d. $y =$



Lesson 19: Four Interesting Transformations of Functions

Classwork

Example 1

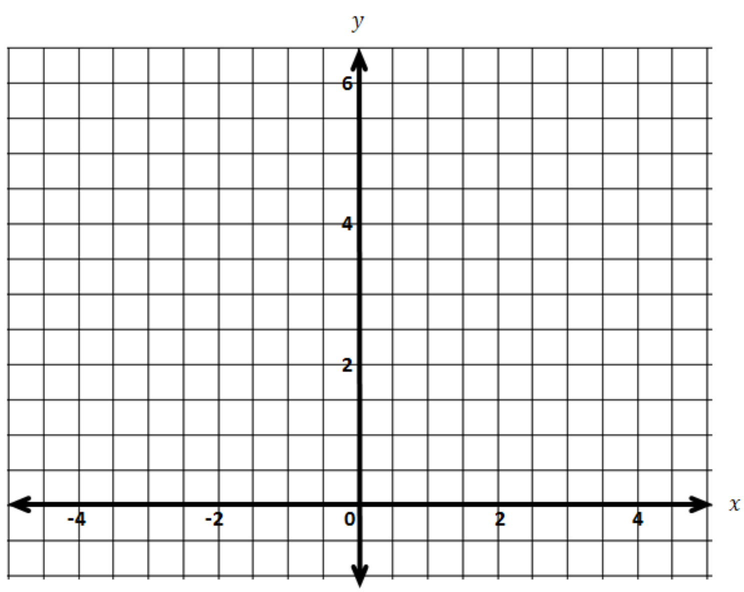
Let $f(x) = x^2$ and $g(x) = f(2x)$, where x can be any real number.

a. Write the formula for g in terms of x^2 (i.e., without using $f(x)$ notation):

b. Complete the table of values for these functions.

x	$f(x) = x^2$	$g(x) = f(2x)$
-3		
-2		
-1		
0		
1		
2		
3		

c. Graph both equations: $y = f(x)$ and $y = f(2x)$.



d. How does the graph of $y = g(x)$ relate to the graph of $y = f(x)$?

e. How are the values of f related to the values of g?

Example 2

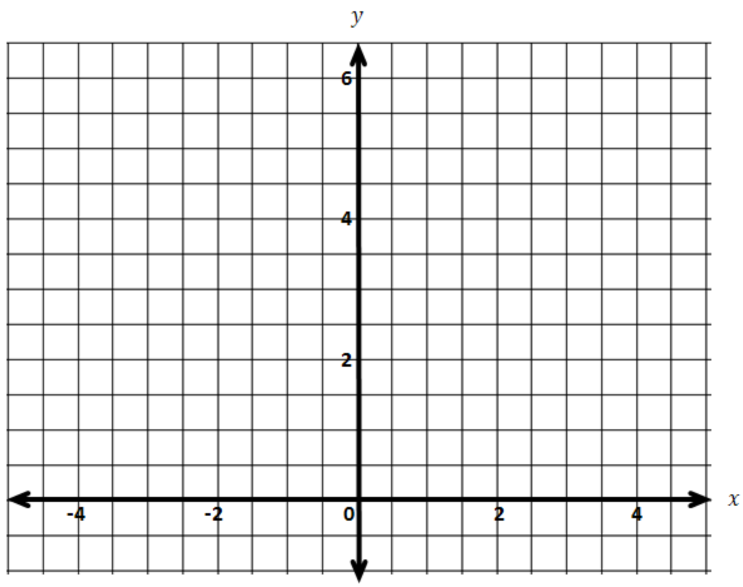
Let $f(x) = x^2$ and $h(x) = f\left(\frac{1}{2}x\right)$, where x can be any real number.

a. Rewrite the formula for h in terms of x^2 (i.e., without using $f(x)$ notation):

b. Complete the table of values for these functions.

x	$f(x) = x^2$	$h(x) = f\left(\frac{1}{2}x\right)$
-3		
-2		
-1		
0		
1		
2		
3		

c. Graph both equations: $y = f(x)$ and $y = f\left(\frac{1}{2}x\right)$.



d. How does the graph of $y = f(x)$ relate to the graph of $y = h(x)$?

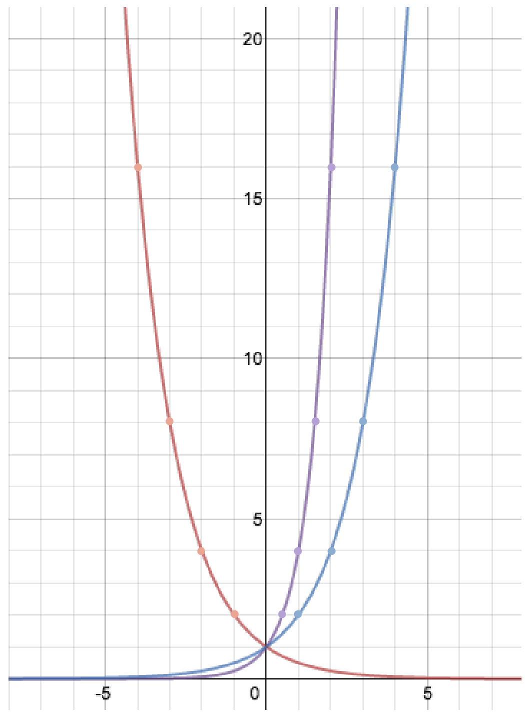
e. How are the values of f related to the values of h?

Exercise 1

Complete the table of values for the given functions.

a.

x	$f(x) = 2^x$	$g(x) = 2^{(2x)}$	$h(x) = 2^{(-x)}$
-2			
-1			
0			
1			
2			



b. Label each of the graphs with the appropriate functions from the table.

c. Describe the transformation that takes the graph of $y = f(x)$ to the graph of $y = g(x)$.

d. Consider $y = f(x)$ and $y = h(x)$. What does negating the input do to the graph of f?

e. Write the formula of an exponential function whose graph would be a horizontal stretch relative to the graph of g.

Example 3

a. Look at the graph of $y = f(x)$ for the function $f(x) = x^2$ in Example 1 again. Would we see a difference in the graph of $y = g(x)$ if -2 was used as the scale factor instead of 2? If so, describe the difference. If not, explain why not.

b. A reflection across the y-axis takes the graph of $y = f(x)$ for the function $f(x) = x^2$ back to itself. Such a transformation is called a *reflection symmetry*. What is the equation for the graph of the reflection symmetry of the graph of $y = f(x)$?

c. Deriving the answer to the following question is fairly sophisticated; do only if you have time: In Lessons 17 and 18, we used the function $f(x) = |x|$ to examine the graphical effects of transformations of a function. Here in Lesson 19, we use the function $f(x) = x^2$ to examine the graphical effects of transformations of a function. Based on the observations you made while graphing, why would using $f(x) = x^2$ be a better option than using the function $f(x) = |x|$?

Lesson 20: Four Interesting Transformations of Functions

Classwork

Opening Exercise

Fill in the blanks of the table with the appropriate heading or descriptive information.

<table>
<tr><th>Graph of $y = f(x)$</th><th colspan="3">Vertical</th><th colspan="3">Horizontal</th></tr>
<tr><td rowspan="2">Translate</td><td rowspan="2">$y = f(x) + k$</td><td>$k > 0$</td><td>Translate up by $|k|$ units</td><td rowspan="2"></td><td>$k > 0$</td><td>Translate right by $|k|$ units</td></tr>
<tr><td></td><td>Translate down by $|k|$ units</td><td>$k < 0$</td><td></td></tr>
<tr><td rowspan="4">Scale by scale factor k</td><td rowspan="4"></td><td>$k > 1$</td><td></td><td rowspan="4">$y = f\left(\frac{1}{k}x\right)$</td><td></td><td>Horizontal stretch by a factor of $|k|$</td></tr>
<tr><td>$0 < k < 1$</td><td>Vertical shrink by a factor of $|k|$</td><td>$0 < k < 1$</td><td></td></tr>
<tr><td></td><td>Vertical shrink by a factor of $|k|$ and reflection over x-axis</td><td>$-1 < k < 0$</td><td>Horizontal shrink by a factor of $|k|$ and reflection across y-axis</td></tr>
<tr><td>$k < -1$</td><td></td><td>$k < -1$</td><td>Horizontal stretch by a factor of $|k|$ and reflection over y-axis</td></tr>
</table>

Example 1

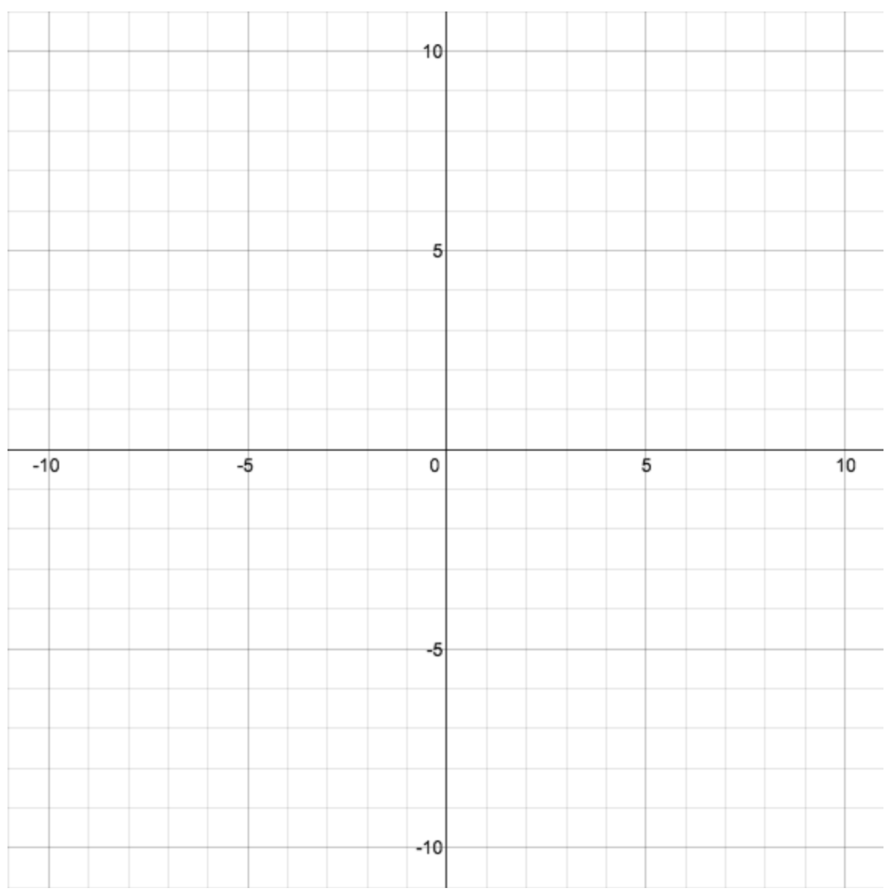
A transformation of the absolute value function, $f(x) = |x - 3|$, is rewritten here as a piecewise function. Describe in words how to graph this piecewise function.

$$f(x) = \begin{cases} -x + 3, & x < 3 \\ x - 3, & x \geq 3 \end{cases}$$

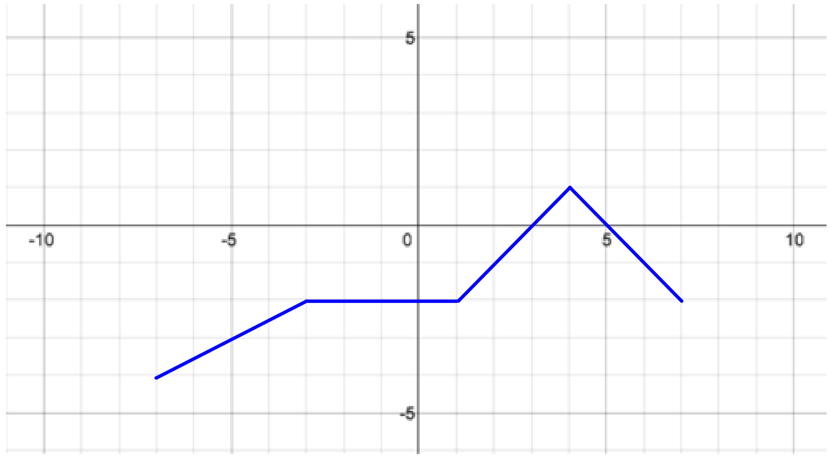
Exercises 1–2

1. Describe how to graph the following piecewise function. Then graph $y = f(x)$ below.

$$f(x) = \begin{cases} -3x - 3, & x \leq -2 \\ 0.5x + 4, & -2 < x < 2 \\ -2x + 9, & x \geq 2 \end{cases}$$



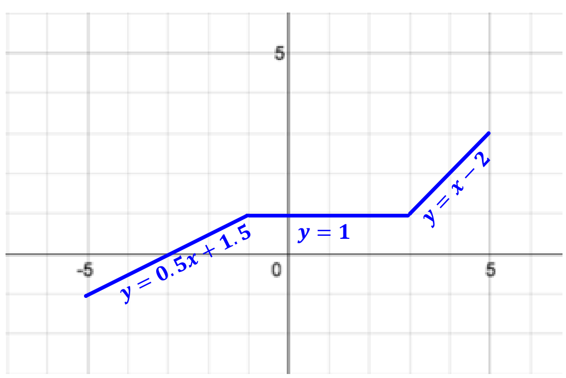
2. Using the graph of f below, write a formula for f as a piecewise function.



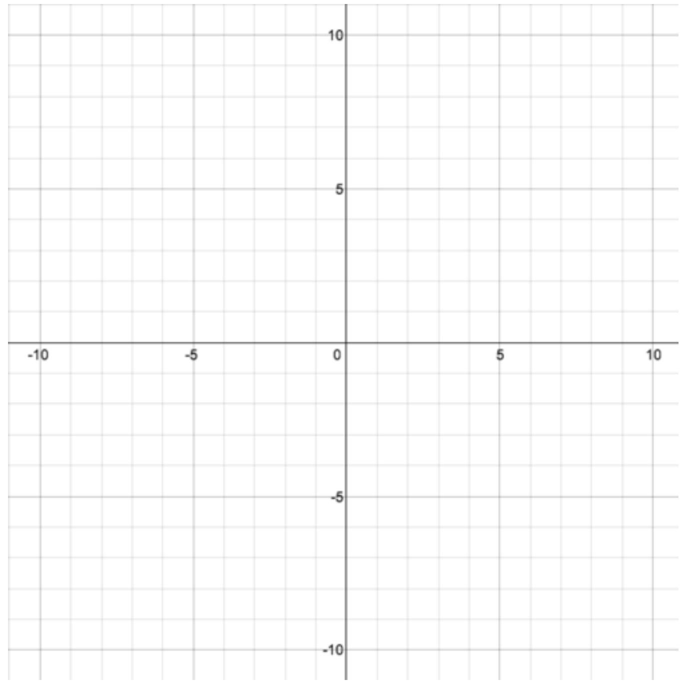
Example 2

The graph $y = f(x)$ of a piecewise function f is shown. The domain of f is $-5 \leq x \leq 5$, and the range is $-1 \leq y \leq 3$.

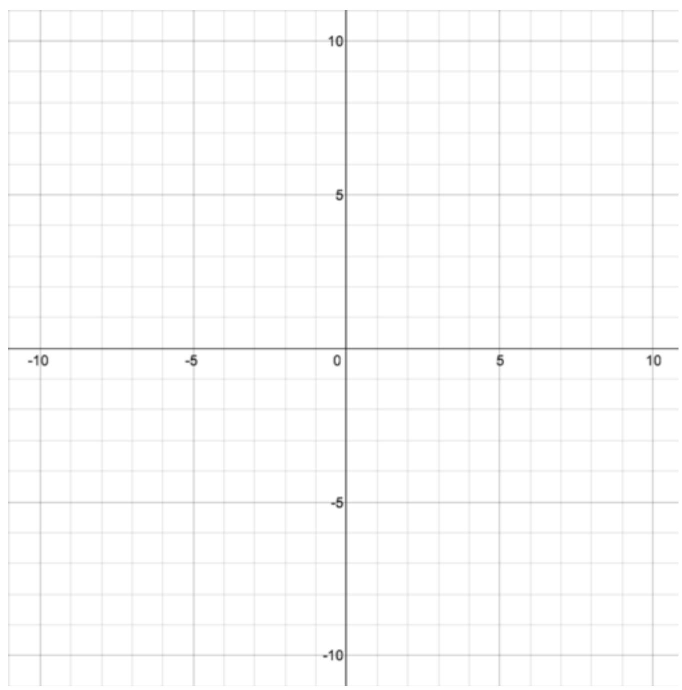
a. Mark and identify four strategic points helpful in sketching the graph of $y = f(x)$.



b. Sketch the graph of $y = 2f(x)$ and state the domain and range of the transformed function. How can you use part (a) to help sketch the graph of $y = 2f(x)$?



c. A horizontal scaling with scale factor $\frac{1}{2}$ of the graph of $y = f(x)$ is the graph of $y = f(2x)$. Sketch the graph of $y = f(2x)$ and state the domain and range. How can you use the points identified in part (a) to help sketch $y = f(2x)$?



Exercises 3–4

3. How does the range of f in Example 2 compare to the range of a transformed function g, where $g(x) = kf(x)$, when $k > 1$?

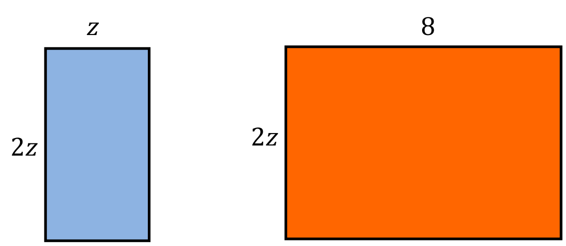
4. How does the domain of f in Example 2 compare to the domain of a transformed function g, where $g(x) = f\left(\frac{1}{k}x\right)$, when $0 < k < 1$? (Hint: How does a graph shrink when it is horizontally scaled by a factor k?)

Lesson 1: Multiplying and Factoring Polynomial Expressions

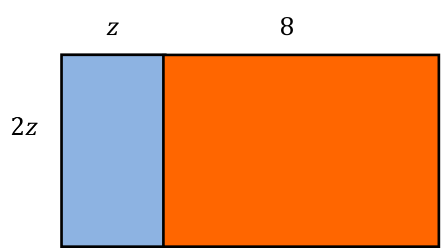
Classwork

Opening Exercise

Write expressions for the areas of the two rectangles in the figures given below.

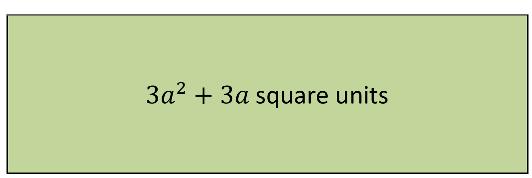


Now write an expression for the area of this rectangle:



Example 1

The total area of this rectangle is represented by $3a^2 + 3a$. Find expressions for the dimensions of the total rectangle.



Exercises 1–3

Factor each by factoring out the Greatest Common Factor:

1. $10ab + 5a$

2. $3g^3h - 9g^2h + 12h$

3. $6x^2y^3 + 9xy^4 + 18y^5$

Discussion: Language of Polynomials

A **prime number** is a positive integer greater than 1 whose only positive integer factors are 1 and itself.

A **composite number** is a positive integer greater than 1 that is not a prime number.

A composite number can be written as the product of positive integers with at least one factor that is not 1 or itself.

For example, the prime number 7 has only 1 and 7 as its factors. The composite number 6 has factors of 1, 2, 3, and 6; it could be written as the product $2 \cdot 3$.

A nonzero polynomial expression with integer coefficients is said to be *prime or irreducible over the integers* if it satisfies two conditions:

1) it is not equivalent to 1 or -1, and
2) if the polynomial is written as a product of two polynomial factors, each with integer coefficients, then one of the two factors must be 1 or -1.

Given a polynomial in standard form with integer coefficients, let c be the greatest common factor of all of the coefficients. The polynomial is *factored completely over the integers* when it is written as a product of c and one or more prime polynomial factors, each with integer coefficients.

Example 2: Multiply Two Binomials

Using a Table as an Aid

You have seen the geometric area model used in previous examples to demonstrate the multiplication of polynomial expressions for which each expression was known to represent a measurement of length.

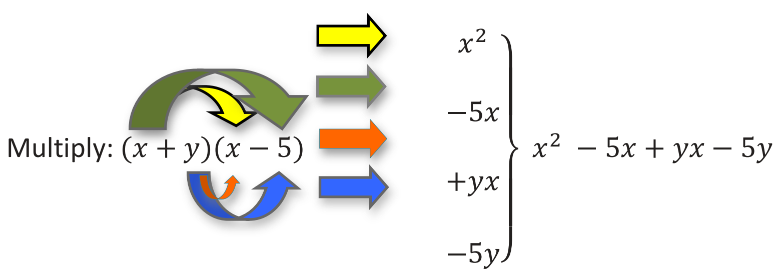
Without a context such as length, we cannot be certain that a polynomial expression represents a positive quantity. Therefore, an area model is not directly applicable to all polynomial multiplication problems. However, a table can be used in a similar fashion to identify each partial product as we multiply polynomial expressions. The table serves to remind us of the area model even though it does not represent area.

For example, fill in the table to identify the partial products of $(x+2)(x+5)$. Then, write the product of $(x+2)(x+5)$ in standard form.

	x	$+\ 5$
x		
$+\ 2$		

Without the Aid of a Table

Regardless of whether or not we make use of a table as an aid, the multiplying of two binomials is an application of the distributive property. Both terms of the first binomial distribute over the second binomial. Try it with $(x+y)(x-5)$. In the example below, the colored arrows match each step of the distribution with the resulting partial product:



Example 3: The Difference of Squares

Find the product of $(x+2)(x-2)$. Use the distributive property to distribute the first binomial over the second.

With the use of a table:

	x	$+\ 2$
x	x^2	$2x$
$+\ -2$	$-2x$	4

$x^2 - 4$

Without the use of a table:

$(x)(x) + (x)(-2) + (2)(x) + (+2)(-2) = x^2 - 2x + 2x - 4 = x^2 - 4$

Exercise 4

Factor the following examples of the difference of perfect squares.

a. $t^2 - 25$

b. $4x^2 - 9$

c. $16h^2 - 36k^2$

d. $4 - b^2$

e. $x^4 - 4$

f. $x^6 - 25$

Write a General Rule for Finding the Difference of Squares

Write $a^2 - b^2$ in factored form.

Exercises 5–7

Factor each of the following differences of squares completely:

5. $9y^2 - 100z^2$

6. $a^4 - b^6$

7. $r^4 - 16s^4$ (Hint: This one will factor twice.)

Example 4: The Square of a Binomial

To square a binomial, such as $(x+3)^2$, multiply the binomial by itself.

$$\begin{aligned}(x+3)(x+3) &= (x)(x)+(3)(x)+(x)(3)+(3)(3)\\ &= x^2+3x+3x+9\\ &= x^2+6x+9\end{aligned}$$

Square the following general examples to determine the general rule for squaring a binomial:

a. $(a+b)^2$

b. $(a-b)^2$

Exercises 8–9

Square the binomial.

8. $(a+6)^2$

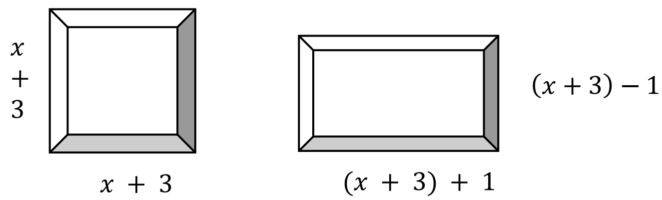
9. $(5-w)^2$

Lesson 3: Advanced Factoring Strategies for Quadratic Expressions

Classwork

Opening Exercise

Carlos wants to build a sandbox for his little brother. He is deciding between a square sandbox with side length that can be represented by $x + 3$ units or a rectangular sandbox with a length 1 unit more than the side of the square and width 1 unit less than the side of the square.



Carlos thinks the areas should be exactly the same since one unit is just moved from one side to the other.

a. Do you agree that the two areas should be the same? Why or why not?

b. How would you write the expressions that represent the length and width of the rectangular sandbox in terms of the side length of the square?

c. If you use the expressions for length and width represented in terms of the side length of the square, can you then write the area of the rectangle in the same terms?

d. How can this expression be seen as the product of a sum and difference, $(a+b)(a-b)$?

e. Can you now rewrite the area expression for the rectangle as the difference of squares: $(a+b)(a-b)=a^2-b^2$?

f. Look carefully at your answer to the last question. What does it tell you about the areas of the two shapes?

g. Can you verify that our algebra is correct using a diagram or visual display?

Exercises 1–6

Factor the expanded form of these quadratic expressions. Pay particular attention to the negative and positive signs.

1. $3x^2 - 2x - 8$

2. $3x^2 + 10x - 8$

3. $3x^2 + x - 14$ [Notice that there is a 1 as a coefficient in this one.]

4. $2x^2 - 21x - 36$ [This might be a challenge. If it takes too long, try the next one.]

5. $-2x^2 + 3x + 9$ [This one has a negative on the leading coefficient.]

6. $r^2 + \frac{6}{4}r + \frac{9}{16}$ [We need to try one with fractions, too.]

Exercises 7–10

Use the structure of these expressions to factor completely.

7. $100x^2 - 20x - 63$

8. $y^4 + 2y^2 - 3$

9. $9x^2 - 3x - 12$

10. $16a^2b^4 + 20ab^2 - 6$

Lesson 5: The Zero Product Property

Classwork

Opening Exercise

Consider the equation $a \cdot b \cdot c \cdot d = 0$. What values of a, b, c, and d would make the equation true?

Exercises 1–4

Find values of c and d that satisfy each of the following equations. (There may be more than one correct answer.)

1. $cd = 0$

2. $(c - 5)d = 2$

3. $(c-5)d=0$

4. $(c-5)(d+3)=0$

Example 1

For each of the related questions below use what you know about the zero product property to find the answers.

a. The area of a rectangle can be represented by the expression, x^2+2x-3. Write each dimension of this rectangle as a binomial, and then write the area in terms of the product of the two binomials.

b. Can we draw and label a diagram that represents the rectangle's area?

c. Suppose the rectangle's area is known to be 21 square units? Can you find the dimensions in terms of x?

d. Rewrite the equation so that it is equal to zero and solve.

e. What are the actual dimensions of the rectangle?

f. A smaller rectangle can fit inside the first rectangle, and it has an area that can be expressed by the equation $x^2 - 4x - 5$. What are the dimensions of the smaller rectangle in terms of x?

g. What value for x would make the smaller rectangle have an area of $\frac{1}{3}$ that of the larger?

Exercises 5–8

Solve. Show your work:

5. $x^2 - 11x + 19 = -5$

6. $7x^2 + x = 0$

7. $7r^2 - 14r = -7$

8. $2d^2 + 5d - 12 = 0$

Lesson 11: Completing the Square

Classwork

Opening Exercise

Rewrite the following perfect square quadratic expressions in standard form. Look for patterns in the coefficients and write two sentences describing what you notice.

FACTORED FORM	WRITE THE FACTORS	DISTRIBUTE	STANDARD FORM
Example: $(x+1)^2$			
$(x+2)^2$			
$(x+3)^2$			
$(x+4)^2$			
$(x+5)^2$			
$(x+20)^2$			

Example 1

Now try working backwards. Rewrite the following standard form quadratic expressions as perfect squares.

STANDARD FORM	FACTORED FORM
$x^2+12x+36$	
$x^2-12x+36$	
$x^2+20x+100$	
$x^2-3x+\frac{9}{4}$	
$x^2+100x+2{,}500$	
x^2+8x+3	

Example 2

Find an expression equivalent to $x^2 + 8x + 3$ that includes a perfect square binomial.

Exercises 1–10

Rewrite each expression by completing the square.

1. $a^2 - 4a + 15$

2. $n^2 - 2n - 15$

3. $c^2 + 20c - 40$

4. $x^2 - 1{,}000x + 60{,}000$

5. $y^2 - 3y + 10$

6. $k^2 + 7k + 6$

7. $z^2 - 0.2z + 1.5$

8. $p^2 + 0.5p + 0.1$

9. $j^2 - \frac{3}{4}j + \frac{3}{4}$

10. $x^2 - bx + c$

Lesson 17: Graphing Quadratic Functions from the Standard Form, $f(x) = ax^2 + bx + c$

Classwork

Opening Exercise

A high school baseball player throws a ball straight up into the air for his math class. The math class was able to determine that the relationship between the height of the ball and the time since it was thrown could be modeled by the function, $h(t) = -16t^2 + 96t + 6$, where t represents the time (in seconds) since the ball was thrown and h, the height (in feet) of the ball above the ground.

a. What does the domain of the function represent in this context?

b. What does the range of this function represent?

c. At what height does the ball get thrown?

d. After how many seconds does the ball hit the ground?

e. What is the maximum height that the ball reaches while in the air? How long will the ball take to reach its maximum height?

f. What feature(s) of this quadratic function are "visible" since it is presented in the standard form, $f(x) = ax^2 + bx + c$?

g. What feature(s) of this quadratic function are "visible" when it is rewritten in vertex form, $f(x) = a(x - h)^2 + k$?

A general strategy for graphing a quadratic function from the standard form:

Example 1

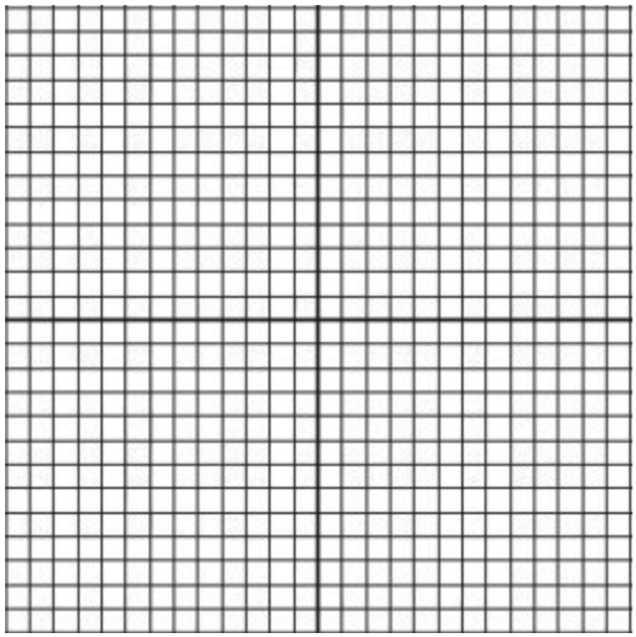
A high school baseball player throws a ball straight up into the air for his math class. The math class was able to determine that the relationship between the height of the ball and the time since it was thrown could be modeled by the function, $h(t) = -16t^2 + 96t + 6$, where t represents the time (in seconds) since the ball was thrown and h, the height (in feet) of the ball above the ground.

a. What do you notice about the equation, just as it is, that will help us in creating our graph?

b. Can we factor to find the zeros of the function? If not, solve $h(t) = 0$ by completing the square.

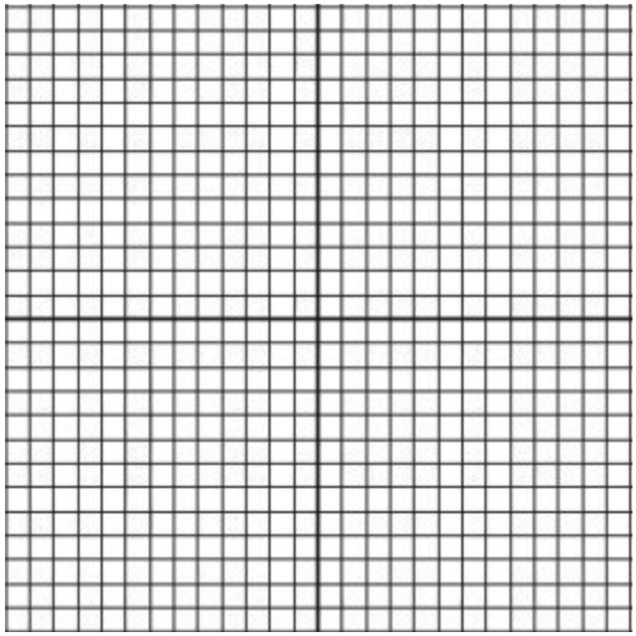
c. Which will you use to find the vertex? Symmetry? Or the completed-square form of the equation?

d. Now we plot the graph of $h(t) = -16t^2 + 96t + 6$ and identify the key features in the graph.

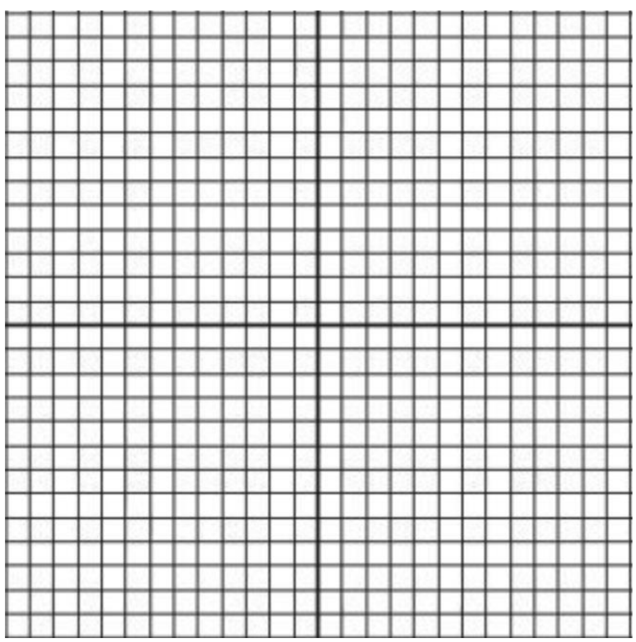


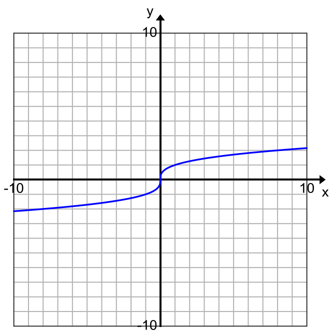
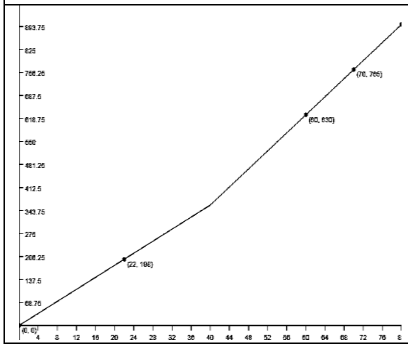
Exercises 1–5

1. Graph the equation $n(x) = x^2 - 6x + 5$ and identify the key features.



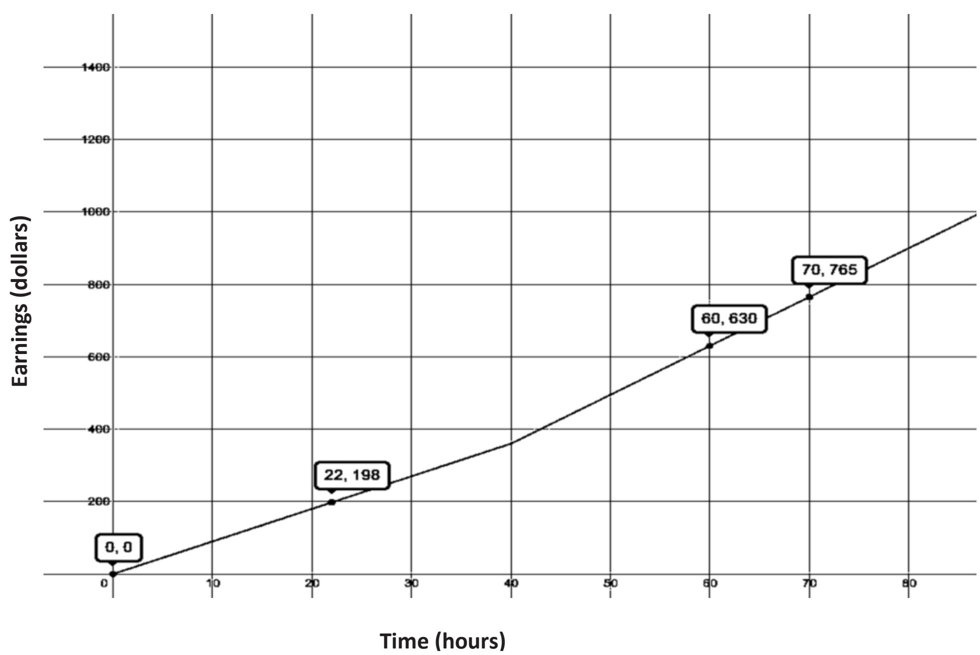
2. Graph the equation $f(x) = \frac{1}{2}x^2 + 5x + 6$ and identify the key features.



Example 1

Eduardo has a summer job that pays him a certain rate for the first 40 hours each week and time-and-a-half for any overtime hours. The graph below shows how much money he earns as a function of the hours he works in one week.



Exercises

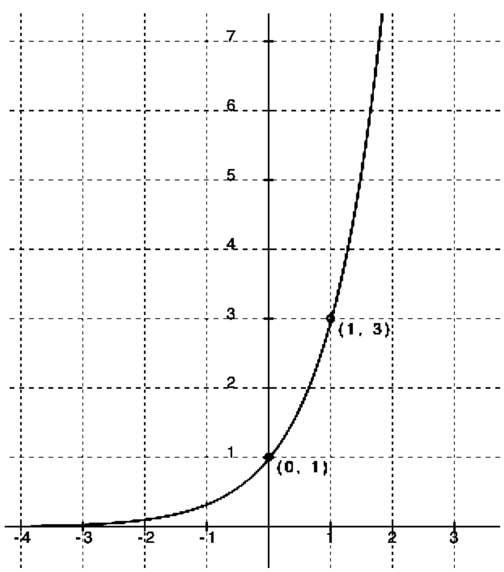
1. Write the function in analytical (symbolic) form for the graph in Example 1.
   a. What is the equation for the first piece of the graph?

   b. What is the equation for the second piece of the graph?

   c. What are the domain restrictions for the context?

   d. Explain the domain in the context of the problem.

For each graph below use the questions and identified ordered pairs to help you formulate an equation to represent it.

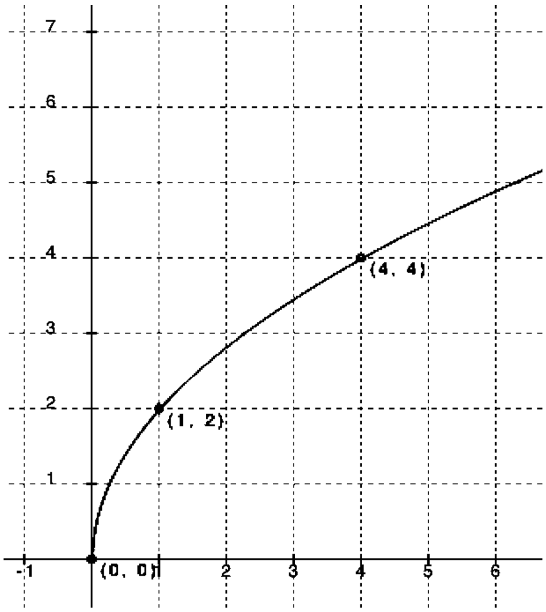


2. Function type:

   Parent function:

   Transformations:

   Equation:

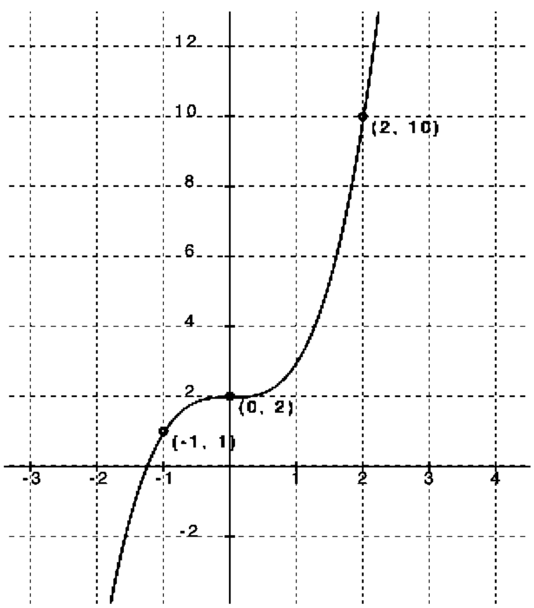


3. Function type:

   Parent function:

   Transformations:

   Equation:

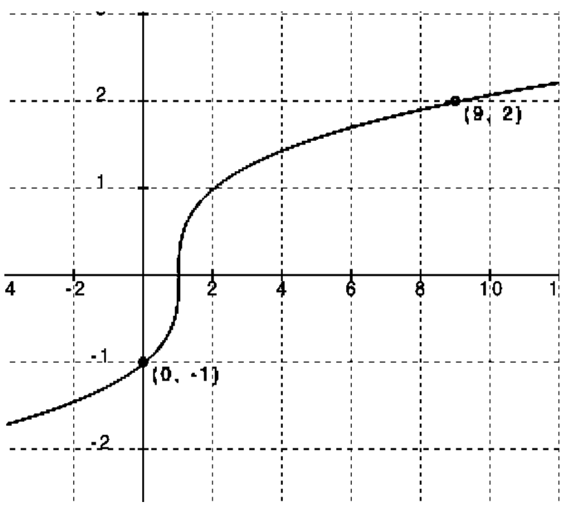


4. Function type:

   Parent function:

   Transformations:

   Equation:

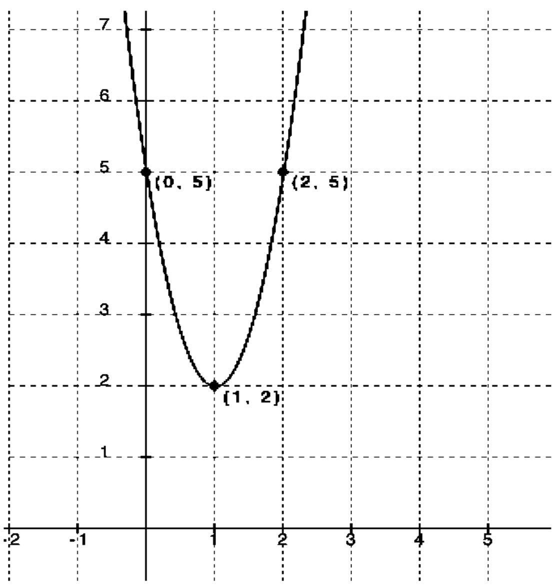


5. Function type:

   Parent function:

   Transformations:

   Equation:



6. Function type:

   Parent function:

   Transformations:

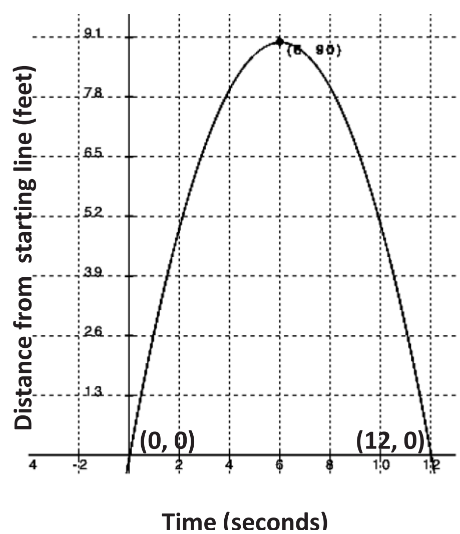
   Equation:

Lesson Summary

- When given a context represented graphically, you first need to:
  - *Identify the variables in the problem (dependent and independent), and*
  - *Identify the relationship between the variables that are described in the graph/situation.*
- To come up with a modeling expression from a graph, you must recognize the type of function the graph represents, observe key features of the graph (including restrictions on the domain), identify the quantities and units involved, and create an equation to analyze the graphed function.
- Identifying a parent function and thinking of the transformation of the parent function to the graph of the function can help with creating the analytical representation of the function.

Problem Set

1. During tryouts for the track team, Bob is running 90-foot wind sprints by running from a starting line to the far wall of the gym and back. At time $t = 0$, he is at the starting line and ready to accelerate toward the opposite wall. As t approaches 6 seconds, he must slow down, stop for just an instant to touch the wall, turn around, and sprint back to the starting line. His distance, in feet, from the starting line with respect to the number of seconds that has passed for one repetition is modeled by the graph below.



   a. What are the key features of this graph?

   b. What are the units involved?

   c. What is the parent function of this graph?

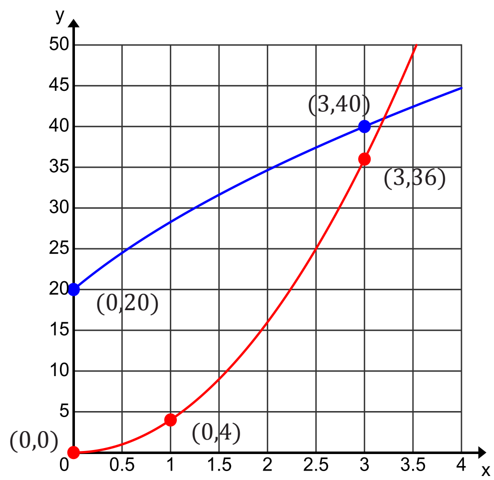
   d. Were any transformations made to the parent functions to get this graph?

   e. What general analytical representation would you expect to model this context?

   f. What do you already know about the parameters of the equation?

   g. Use the ordered pairs you know to replace the parameters in the general form of your equation with constants so that the equation will model this context. Check your answer using the graph.

2. Spencer and McKenna are on a long-distance bicycle ride. Spencer leaves one hour before McKenna. The graph below shows each rider's distance in miles from his or her house as a function of time since McKenna left on her bicycle to catch up with Spencer. (Note: Parts (e), (f), and (g) are challenge problems.)



a. Which function represents Spencer's distance? Which function represents McKenna's distance? Explain your reasoning.

b. Estimate when McKenna catches up to Spencer. How far have they traveled at that point in time?

c. One rider is speeding up as time passes and the other one is slowing down. Which one is which, and how can you tell from the graphs?

d. According to the graphs, what type of function would best model each rider's distance?

e. Create a function to model each rider's distance as a function of the time since McKenna started riding her bicycle. Use the data points labeled on the graph to create a precise model for each rider's distance.

f. What is the meaning of the x- and y-intercepts of each rider in the context of this problem?

g. Estimate which rider is traveling faster 30 minutes after McKenna started riding. Show work to support your answer.

Lesson 2: Analyzing a Data Set

Classwork

Opening Exercise

When tables are used to model functions, we typically have just a few sample values of the function and therefore have to do some detective work to figure out what the function might be. Look at these three tables:

x	$f(x)$
0	6
1	12
2	18
3	24
4	30
5	36

x	$g(x)$
0	0
1	14
2	24
3	30
4	32
5	30

x	$h(x)$
0	1
1	3
2	9
3	27
4	81
5	243

Example 1

Noam and Athena had an argument about whether it would take longer to get from NYC to Boston and back by car or by train. To settle their differences, they made separate, non-stop round trips from NYC to Boston. On the trip, at the end of each hour, both recorded the number of miles they had traveled from their starting points in NYC. The tables below show their travel times, in hours, and the distances from their starting points, in miles. The first table shows Noam's travel time/distance from the starting point, and the second represents Athena's. Use *both* data sets to justify your answers to the questions below.

Time in Hours	Noam's Distance
0	0
1	55
2	110
3	165
4	220
5	165
6	110
7	55
8	0

Time in Hours	Athena's Distance
0	0
1	81
2	144
3	189
4	216
5	225
6	216
7	189
8	144
9	81
10	0

a. Who do you think is driving, and who is riding the train? Explain your answer in the context of the problem.

b. According to the data, how far apart are Boston and New York City? Explain mathematically.

c. How long did it take each of them to make the round trip?

d. According to their collected data, which method of travel was faster?

e. What was the average rate of change for Athena for the interval from 3 to 4 hours? How might you explain that in the context of the problem?

f. Noam believes a quadratic function can be used as a model for both data sets. Do you agree? Use and describe the key features of the functions represented by the data sets to support your answer.

Exercises

1. Explain why each function can or cannot be used to model the given data set:

x	$f(x)$
0	5
1	8
2	9
3	8
4	5
5	0
6	–7

a. $f(x) = 3x + 5$

b. $f(x) = -(x-2)^2 + 9$

c. $f(x) = -x^2 + 4x - 5$

d. $f(x) = 3^x + 4$

e. $f(x) = (x+2)^2 - 9$

f. $f(x) = -(x+1)(x-5)$

2. Match each table below to the function and the context, and explain how you made your decision.

A

x	y
1	9
2	18
3	27
4	18
5	9

Equation ____

Context ____

B

x	y
1	12
2	24
3	36
4	48
5	60

Equation ____

Context ____

C

x	y
0	160
1	174
2	156
3	106
4	24

Equation ____

Context ____

D

x	y
1	2
2	4
3	8
4	16
5	32

Equation ____

Context ____

E

x	y
2	8
3	9
4	8
5	5
6	0

Equation ____

Context ____

Equations:

$f(x) = 12x$

$h(x) = -9|x - 3| + 27$

$g(x) = -(x)(x - 6)$

$p(x) = 2^x$

$q(x) = -16x^2 + 30x + 160$

Contexts:

1. The population of bacteria doubled every month, and the total population vs. time was recorded.
2. A ball was launched upward from the top of a building, and the vertical distance of the ball from the ground vs. time was recorded.
3. The height of a certain animal's vertical leap was recorded at regular time intervals of one second; the animal returned to ground level after six seconds.
4. Melvin saves the same amount of money every month. The total amount saved after each month was recorded.
5. Chris ran at a constant rate on a straight-line path and then returned at the same rate. His distance from his starting point was recorded at regular time intervals.

Lesson Summary

The following methods can be used to determine the appropriate model for a given data set as a linear, quadratic or exponential function:

- If the first difference is constant, then the data set could be modeled by a linear function.
- If the second difference is constant, then the data set could be modeled by a quadratic function.
- If the subsequent y-values are multiplied by a constant, then the data set could be modeled by an exponential function.

Problem Set

1.
   a. Determine the function type that could be used to model the data set at the right and explain why.

   b. Complete the data set using the special pattern of the function you described above.

   c. If it exists, find the minimum or maximum value for the function model. If there is no minimum or maximum, explain why.

x	y
0	
1	10
2	0
3	-6
4	-8
5	
6	

2.
   a. Determine the function type that could be used to model the data set and explain why.

   b. Complete the data set using the special pattern of the function you described above.

   c. If it exists, find the minimum or maximum value for the function model. If there is no minimum/maximum, explain why.

x	y
-1	
0	
1	
2	16
3	64
4	256
5	1024

3.
   a. Determine the function type that could be used to model the data set and explain why.

   b. Complete the data set using the special pattern of the function you described above.

   c. If it exists, find the minimum or maximum value for the function model. If there is no minimum/maximum, explain why.

x	y
-1	
0	12
1	
2	24
3	
4	36
5	

4. Circle all the function types that could possibly be used to model a context if the given statement applies.

   a. When x-values are at regular intervals, the first difference of y-values is not constant.

   Linear Function Quadratic Function Exponential Function Absolute Value Function

   b. The second difference of data values is not constant.

   Linear Function Quadratic Function Exponential Function Absolute Value Function

   c. When x-values are at regular intervals, the quotient of any two consecutive y-values is a constant that is not equal to 0 or 1.

   Linear Function Quadratic Function Exponential Function Absolute Value Function

   d. There may be two different x-values for $y = 0$.

   Linear Function Quadratic Function Exponential Function Absolute Value Function

Lesson 3: Analyzing a Verbal Description

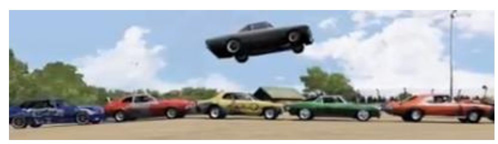
Classwork

Read the example problems below and discuss a problem solving strategy with a partner or small group.

Example 1

Gregory plans to purchase a video game player. He has $\$500$ in his savings account and plans to save $\$20$ per week from his allowance until he has enough money to buy the player. He needs to figure out how long it will take. What type of function should he use to model this problem? Justify your answer mathematically.

Example 2



One of the highlights in a car show event is a car driving up a ramp and "flying" over approximately five cars placed end-to-end. The ramp is 8 feet at its highest point, and there is an upward speed of 88 feet per second before it leaves the top of the ramp. What type of function can best model the height, h, in feet, of the car t seconds after leaving the end of the ramp? Justify your answer mathematically.

Example 3

Margie got $\$1000$ from her grandmother to start her college fund. She is opening a new savings account and finds out that her bank offers a 2% annual interest rate, compounded monthly. What type of function would best represent the amount of money in Margie's account? Justify your answer mathematically.

Exercises

1. City workers recorded the number of squirrels in a park over a period of time. At the first count, there were 15 pairs of male and female squirrels (30 squirrels total). After 6 months, the city workers recorded a total of 60 squirrels, and after a year, there were 120.

   a. What type of function can best model the population of squirrels recorded over a period of time, assuming the same growth rate and that no squirrel dies?

   b. Write a function that represents the population of squirrels recorded over x number of years. Explain how you determined your function.

2. A rectangular photograph measuring 8 inches by 10 inches is surrounded by a frame with a uniform width, x.

   a. What type of function can best represent the area of the picture and the frame in terms of x (the unknown frame's width)? Explain mathematically how you know.

   b. Write an equation in standard form representing the area of the picture and the frame. Explain how you arrive at your equation.

3. A ball is tossed up in the air at an initial rate of 50 feet per second from 5 feet off the ground.
   a. What type of function models the height (h, in feet) of the ball after t seconds?

   b. Explain what is happening to the height of the ball as it travels over a period of time (in t seconds).

   c. What function models the height, h (in feet), of the ball over a period of time (in t seconds)?

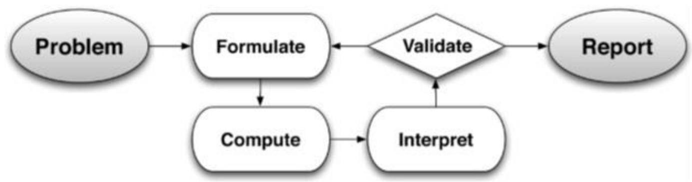
4. A population of insects is known to triple in size every month. At the beginning of a scientific research project, there were 200 insects.
   a. What type of function models the population of the insects after t years?

   b. Write a function that models the population growth of the insects after t years.

Lesson 4: Modeling a Context from a Graph

Classwork

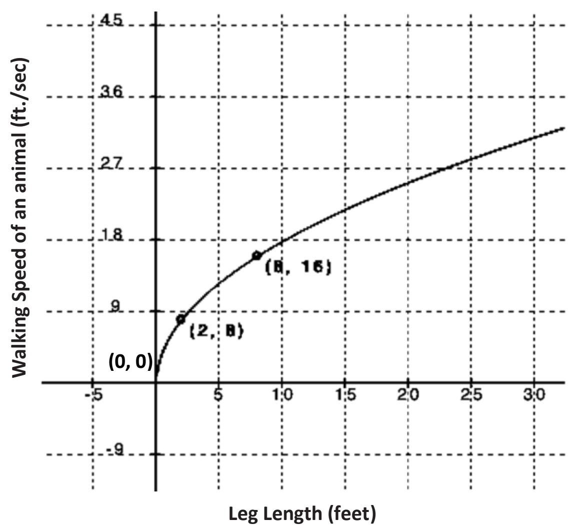
Opening Exercise



Example 1

Read the problem below. Your teacher will walk you through the process of using the steps in the modeling cycle to guide in your solution.

The relationship between the length of one of the legs, in feet, of an animal and its walking speed, in feet per second, can be modeled by the graph below. [Note: This function applies to *walking* not *running* speed. Obviously, a cheetah has shorter legs than a giraffe but can run much faster. However, in a walking race, the giraffe has the advantage.]

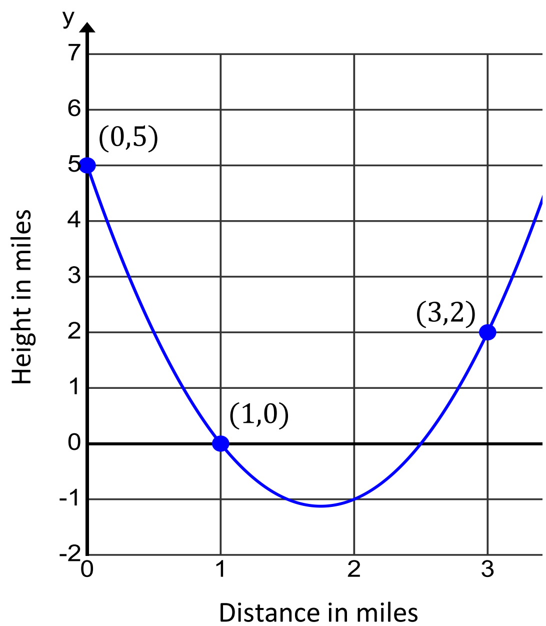


A T-Rex's leg length was 20 ft. What was the T-Rex's speed in ft./sec.?

d. VALIDATE

i. How can you check to make sure your function models the graph accurately?

2. The cross-section view of a deep river gorge is modeled by the graph shown below where both height and distance are measured in miles. How long is a bridge that spans the gorge from the point labeled $(1,0)$ to the other side? How high above the bottom of the gorge is the bridge?



a. FORMULATE

i. What type of function can be represented by a graph like this? (Linear, quadratic, exponential, piecewise, square root, or cube root)

ii. What are the quantities in this problem?

iii. How would you describe the end behavior of the graph?

iv. What is a general form for this function type?

v. How does knowing the function type and end behavior affect the equation of the function for this graph?

vi. What is the equation we would use to model this graph?

b. COMPUTE

i. What are the key features of the graph that can be used to determine the equation?

ii. Which key features of the function must be determined?

iii. Calculate the missing key features and check for accuracy with your graph.

c. INTERPRET

i. What domain makes sense for this context? Explain.

ii. How wide is the bridge with one side located at $(1,0)$?

iii. How high is the bridge above the bottom of the gorge?

iv. Suppose the gorge is exactly 3.5 feet wide from its two highest points. Find the average rate of change for the interval from $x = 0$ to $x = 3.5$, $[0, 3.5]$. Explain this phenomenon. Are there other intervals that will behave similarly?

d. VALIDATE

i. How can you check to make sure that your function models the graph accurately?

Now compare four representations that may be involved in the modeling process. How is each useful for each phase of the modeling cycle? Explain the advantages and disadvantages of each.

Lesson 4: Modeling a Context from a Graph

Name ____ Date____________________

Exit Ticket

1. Why might we want to represent a graph of a function in analytical form?

2. Why might we want to represent a graph as a table of values?

Exit Ticket Sample Solutions

1. **Why might we want to represent a graph of a function in analytical form?**

   Graphs require estimation for many values, and for most we can calculate exact values using the function equation. Some key features that may not be visible or clear on a graph can be seen in the symbolic representation.

2. **Why might we want to represent a graph as a table of values?**

   In a table of values, we can sometimes better see patterns in the relationship between the x- and y-values.

Lesson 5: Modeling from a Sequence

Classwork

Opening Exercise

A soccer coach is getting her students ready for the season by introducing them to High Intensity Interval Training (HIIT). She presents the table below with a list of exercises for an HIIT training circuit and the length of time that must be spent on each exercise before the athlete gets a short time to rest. The rest times increase as the students complete more exercises in the circuit. Study the chart and answer the questions below. How long would the 10^{th} exercise be? If a player had 30 minutes of actual gym time during a period, how many exercises could she get done? Explain your answers.

Exercise #	Length of Exercise Time	Length of Rest Time
Exercise 1	0.5 minutes	0.25 minutes
Exercise 2	0.75 minutes	0.5 minutes
Exercise 3	1 minute	1 minutes
Exercise 4	1.25 minutes	2 minutes
Exercise 5	1.5 minutes	4 minutes

Example 1

Determine whether the sequence below is arithmetic or geometric and find the function that will produce any given term in the sequence:

$$16, 24, 36, 54, 81, \dots$$

Is this sequence arithmetic?

Is the sequence geometric?

What is the analytical representation of the sequence?

Lesson Summary

- A sequence is a list of numbers or objects in a special order.
- An arithmetic sequence goes from one term to the next by adding (or subtracting) the same value.
- A geometric sequence goes from one term to the next by multiplying (or dividing) by the same value.
- Looking at the difference of differences can be a quick way to determine if a sequence can be represented as a quadratic expression.

Problem Set

Solve the following problems by finding the function/formula that represents the n^{th} term of the sequence.

1. After a knee injury, a jogger is told he can jog 10 minutes every day and that he can increase his jogging time by 2 minutes every two weeks. How long will it take for him to be able to jog one hour a day?

Week #	Daily Jog Time
1	10
2	10
3	12
4	12
5	14
6	14

2. A ball is dropped from a height of 10 feet. The ball then bounces to 80% of its previous height with each subsequent bounce.
   a. Explain how this situation can be modeled with a sequence.
   b. How high (*to the nearest tenth of a foot*) does the ball bounce on the fifth bounce?

3. Consider the following sequence:

$$8, 17, 32, 53, 80, 113, \ldots$$

   a. What pattern do you see, and what does that pattern mean for the analytical representation of the function?
   b. What is the symbolic representation of the sequence?

4. Arnold wants to be able to complete 100 military-style pull-ups. His trainer puts him on a workout regimen designed to improve his pull-up strength. The following chart shows how many pull-ups Arnold can complete after each month of training. How many months will it take Arnold to achieve his goal if this pattern continues?

Month	Pull-Up Count
1	2
2	5
3	10
4	17
5	26
6	37
…	

Lesson 6: Modeling a Context from Data

Classwork

Opening Exercise

1. Identify the type of function that each table represents (e.g., quadratic, linear, exponential, square root, etc.).

A

x	y
1	5
2	7
3	9
4	11
5	13

B

x	y
1	6
2	9
3	13.5
4	20.25
5	30.375

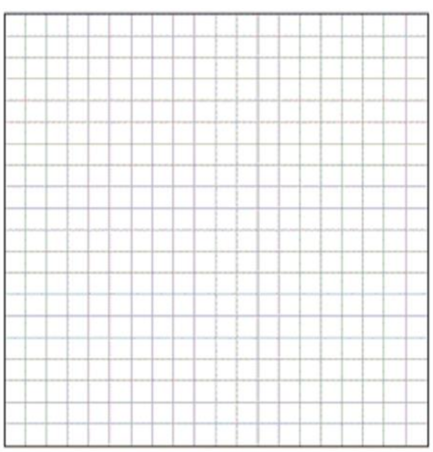
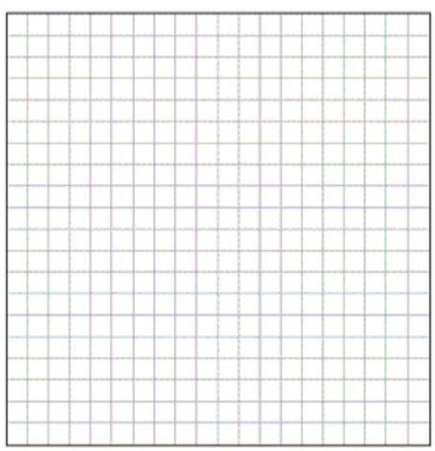
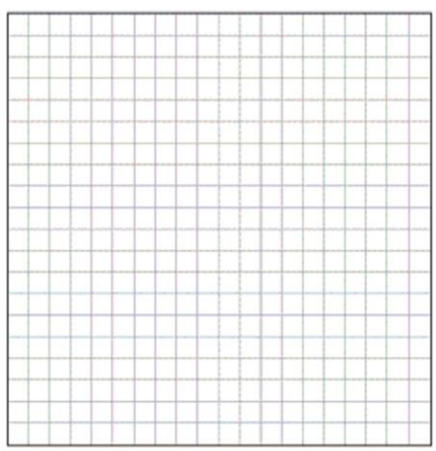
C

x	y
1	3
2	12
3	27
4	48
5	75

2. Explain how you were able to identify the function.

3. Find the symbolic representation of the function.

4. Plot the graphs of your data.

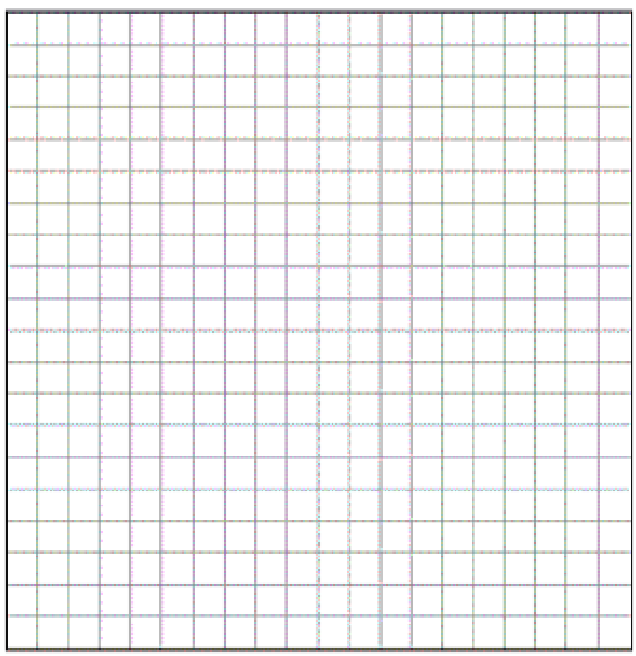




Example 1

Enrique is a biologist who has been monitoring the population of a rare fish in Lake Placid. He has tracked the population for 5 years and has come up with the following estimates:

Year Tracked	Year Since 2002	Estimated Fish Population
2002	0	1000
2003	1	899
2004	2	796
2005	3	691
2006	4	584

Create a graph and a function to model this situation, and use it to predict (assuming the trend continues) when the fish population will be gone from the Lake Placid ecosystem. Verify your results, and explain the limitations of each model.



Exercises

1. Bella is a BMX bike racer and wants to identify the relationship between her bike's weight and the height of jumps (a category she gets judged on when racing). On a practice course, she tests out 7 bike models with different weights and comes up with the following data.

Weight (lbs.)	Height of Jump (ft.)
20	8.9
21	8.82
22	8.74
23	8.66
24	8.58
25	8.5
26	8.42
27	8.34

a. Bella is sponsored by Twilight Bikes and must ride a 32-lb bike. What can she expect her jump height to be?

b. Bella asks the bike engineers at Twilight to make the lightest bike possible. They tell her the lightest functional bike they could make is 10 lbs. Based on this data, what is the highest she should expect to jump if she only uses Twilight bikes?

c. What is the maximum weight of a bike if Bella's jumps have to be at least 2 feet high during a race?

2. The concentration of medicine in a patient's blood as time passes is recorded in the table below.

Time (hours)	Concentration of Medicine (ml)
0	0
0.5	55.5
1	83
1.5	82.5
2	54

a. The patient cannot be active while the medicine is in his blood. How long, to the nearest minute, must the patient remain inactive? What are the limitations of your model(s)?

b. What is the highest concentration of medicine in the patient's blood?

3. A student is conducting an experiment, and as time passes, the number of cells in the experiment decreases. How many cells will there be after 16 minutes?

Time (minutes)	Cells
0	$5{,}000{,}000$
1	$2{,}750{,}000$
2	$1{,}512{,}500$
3	$831{,}875$
4	$457{,}531$
5	$251{,}642$
6	$138{,}403$

Lesson Summary

When given a data set, strategies that could be used to determine the type of function that describes the relationship between the data are

- Determine the variables involved and plot the points.
- After making sure the x-values are given at regular intervals, look for common differences between the data points – first and second.
- Determine the type of sequence the data models first, and then use the general form of the function equation to find the parameters for the symbolic representation of the function.

Lesson 6: Modeling a Context from Data

Exit Ticket

1. Lewis's dad put $1,000$ dollars in a money market fund with a fixed interest rate when he was 16. Lewis can't touch the money until he is 26, but he gets updates on the balance of his account.

Years After Lewis Turns 16	Account Balance in Dollars
0	1000
1	1100
2	1210
3	1331
4	1464

a. Develop a model for this situation.

b. Use your model to determine how much Lewis will have when he turns 26 years old.

c. Comment on the limitations/validity of your model.

Exit Ticket Sample Solutions

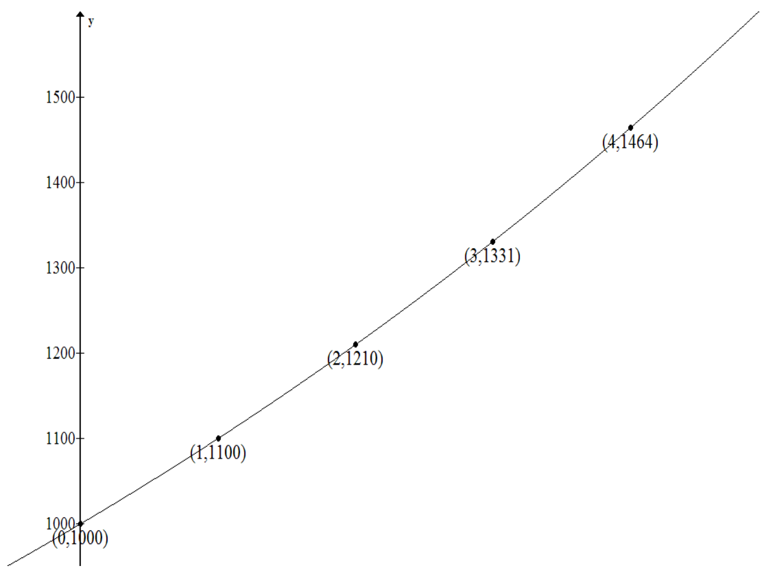
1. Lewis's dad put $1,000$ dollars in a money market fund when he was 16. Lewis can't touch the money until he is 26, but he gets updates on the balance of his account.

Years After Lewis Turns 16	Account Balance in Dollars
0	1000
1	1100
2	1210
3	1331
4	1464

a. Develop a model for this situation.

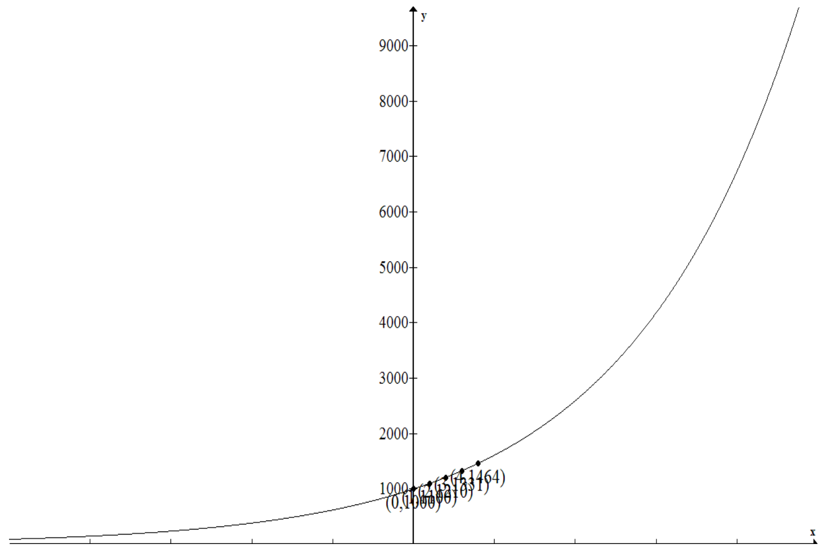
We might try graphing this data. However, in the viewing window that shows our data points (see graph below), it appears that the function might be linear. Let's try zooming out to see more of the key features of this graph. (See graph below.)

x: $[0, 5]$ y: $[975, 1500]$



This viewing window gives us a close-up of the data points and their relation to each other. However, we cannot really see the features of the graph that represent the data.

x: $[-25, 25]$ y: $[0, 9700]$



In this version of the graph, you can see how the data from our table is grouped on a very short section of the graph. From this view, we can see the exponential nature of the graph.

Using the data table to find a function model: The first and second differences have no commonalities; therefore, this is not linear or quadratic. Checking to see if there is a common ratio, we see that this is an exponential relationship (or a geometric sequence) where the common ratio is 1.1, and the initial value is 1000. Check: Since on this table time starts at $t = 0$, using t as the exponent will yield $\$1000$ for the initial balance.

Therefore, we can represent this sequence symbolically by: $A(t) = 1000(1.1)^t$

b. Use your model to determine how much Lewis will have when he turns 26 years old.

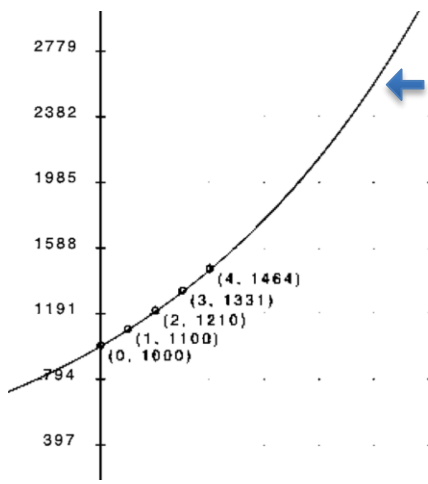
Using the function: Since Lewis will be 26 ten years after he turns 16, we will need to evaluate $A(10)$:

$$A(10) = 1000(1.1)^{10} \rightarrow A(10) = 1000(1.1)^{10} \rightarrow A(10) = 2593.74 \text{ or } \$2,593.74$$

We might also try extending our data table to verify this result. There are a couple of precision decisions to make: Shall we use 1.1 as the common ratio? How soon should we begin rounding numbers off? For this table, we decided to use 1.1 as the common ratio and did not round until the final step.

Lewis' Age	Years After Lewis Turns 16	Account Balance in Dollars
16	0	1000
17	1	1100
18	2	1210
19	3	1331
20	4	1464.1
21	5	1610.51
22	6	1771.56
23	7	1948.72
24	8	2143.59
25	9	2357.95
26	10	2593.74

We might also try to answer this question using our graph. Below is another view of the graph. Can you estimate the balance at $t = 10$?



c. **Comment on the limitations/validity of your model.**

As we saw in the first and second versions of the graph, there are limitations to the graphic model because we cannot always see the key features of the graph in a window that lets us see all the data points clearly. Being able to see the graph using both windows was more helpful. Then, in part (b), we saw how difficult it was to estimate the value of the function at $t = 10$ for such large numbers of $A(t)$. We were also able to extend the table without too much difficulty, after deciding what level of precision we needed to use.

The equation was most helpful but requires interpretation of the data (noticing that the common ratio was very close but not absolutely perfect, and making sure we started with $t = 0$).

Regardless of whether we use a graphical, numerical, or algebraic model, one limitation is that we are assuming the growth rate will remain constant until he is 26.

Lesson 8: Modeling a Context from a Verbal Description

Classwork

Example 1

Christine has $\$500$ to deposit in a savings account, and she is trying to decide between two banks. Bank A offers 10% annual interest compounded quarterly. Rather than compounding interest for smaller accounts, Bank B offers to add $\$15$ quarterly to any account with a balance of less than $\$1000$ for every quarter, as long as there are no withdrawals. Christine has decided that she will neither withdraw, nor make a deposit for a number of years.

Develop a model that will help Christine decide which bank to use.

Example 2

Alex designed a new snowboard. He wants to market it and make a profit. The total initial cost for manufacturing set-up, advertising, etc. is $\$500{,}000$, and the materials to make the snowboards cost $\$100$ per board.

The demand function for selling a similar snowboard is $D(p) = 50{,}000 - 100p$, where p = selling price of each snowboard.

a. Write an expression for each of the following. Let p represent the selling price:

Demand Function (number of units that will sell)

Revenue (number of units that will sell, price per unit, p)

Total Cost (cost for producing the snowboards)

b. Write an expression to represent the profit.

c. What is the selling price of the snowboard that will give the maximum profit?

d. What is the maximum profit Alex can make?

Exercises

Alvin just turned 16 years old. His grandmother told him that she will give him $\$10{,}000$ to buy any car he wants whenever he is ready. Alvin wants to be able to buy his dream car by his 21st birthday, and he wants a 2009 Avatar Z, which he could purchase today for $\$25{,}000$. The car depreciates (reduces in value) at a rate is 15% per year. He wants to figure out how long it would take for his $\$10{,}000$ to be enough to buy the car, without investing the $10,000.

1. Write the function that models the depreciated value of the car after n number of years.

After n years	Value of the Car
1	
2	
3	
4	
5	
6	

a. Will he be able to afford to buy the car when he turns 21? Explain why or why not.

b. Given the same rate of depreciation, after how many years will the value of the car be less than $\$5{,}000$?

Lesson Summary

- We can use the full modeling cycle to solve real-world problems in the context of business and commerce (e.g., compound interest, revenue, profit, and cost) and population growth and decay (e.g., population growth, depreciation value, and half-life) to demonstrate linear, exponential and quadratic functions described verbally through using graphs, tables, or algebraic expressions to make appropriate interpretation and decision.
- Sometimes a graph or table is the best model for problems that involve complicated function equations.

Problem Set

1. Maria invested $\$10{,}000$ in the stock market. Unfortunately, the value of her investment has been dropping at an average rate of 3% each year.
   a. Write the function that best models the situation.
   b. If the trend continues, how much will her investment be worth in 5 years? (For $n = 5$)
   c. Given the situation, what should she do with her investment?

2. The half-life of the radioactive material in *Z-Med*, a medication used for certain types of therapy, is 2 days. A patient receives a 16-mCi dose (millicuries, a measure of radiation) in his treatment. [Half-life means that the radioactive material decays to the point where only half is left.]
   a. Make a table to show the level of *Z-Med* in the patient's body after n days.

Number of days	Level of *Z-Med* in patient
0	
2	
4	
6	
8	
10	

   b. Write an equation for $f(n)$ to model the half-life of *Z-Med* for n days. [Be careful here. Make sure that the formula works for both odd and even numbers of days.]
   c. How much radioactive material from *Z-Med* is left in the patient's body after 20 days of receiving the medicine?